

RATIONAL CURVES IN CALABI-YAU THREEFOLDS

TRYGVE JOHNSEN AND ANDREAS LEOPOLD KNUTSEN

Address: Dept. of Mathematics,
University of Bergen, Johs. Bruns gt 12,
N-5008 Bergen, Norway

E-mail: johnsen@mi.uib.no, andreask@mi.uib.no

This paper is dedicated to Steven L. Kleiman on his 60th birthday

ABSTRACT. We study the set of rational curves of a certain topological type in general members of certain families of Calabi-Yau threefolds. For some families we investigate to what extent it is possible to conclude that this set is finite. For other families we investigate whether this set contains at least one point representing an isolated rational curve. Our study is inspired by [Jo-K11].

1. INTRODUCTION

The famous Clemens conjecture says roughly that for each fixed d there is only a finite, but non-empty set of rational curves of degree d on a general quintic threefold F in complex \mathbf{P}^4 . A more ambitious version is the following:

The Hilbert scheme of rational, smooth and irreducible curves C of degree d on a general quintic threefold in \mathbf{P}^4 is finite, nonempty and reduced, so each curve is embedded with balanced normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Katz proved this statement for $d \leq 7$, and in [Nj] and [Jo-K11] the result was extended to $d \leq 9$. An even more ambitious version also includes the statement that for general F , there are no singular rational curves of degree d , and no intersecting pair of rational curves of degrees d_1 and d_2 with $d_1 + d_2 = d$. This was proven (in [Jo-K11])

1991 *Mathematics Subject Classification.* 14J32 (14H45, 14J28).

Key words and phrases. rational curves, Calabi-Yau threefolds.

to hold for $d \leq 9$, with the important exception for $d = 5$, where a general quintic contains a finite number of 6-nodal plane curves. This number was computed in [Va].

We see that in rough terms the conjecture contains a finiteness part and an existence part (existence of at least one isolated smooth rational curve of fixed degree d on general F , for each natural number d). For the quintics in \mathbf{P}^4 , the existence part was proved for all d by S. Katz [Ka], extending an argument from [Cl1], where existence was proved for infinitely many d .

In this paper we will sum up or study how the situation is for some concrete families of embedded Calabi-Yau threefolds other than the quintics in \mathbf{P}^4 .

In Section 2 we will briefly sketch some finiteness results for Calabi-Yau threefolds that are complete intersections in projective spaces. There are four other families of Calabi-Yau threefolds F that are such complete intersections, namely those of type (2,4) and (3,3) in \mathbf{P}^5 , those of type (2,2,3) in \mathbf{P}^6 , and those of type (2,2,2,2) in \mathbf{P}^7 . For these families we have finiteness results comparable to those for quintics in \mathbf{P}^4 involving smooth curves. The existence question has been answered in positive terms for curves of low genera, including $g = 0$ and all positive d , in these cases. See [Kl2] and [E-J-S].

In Section 3 we study the five other families of Calabi-Yau threefolds F that are *complete intersections with Grassmannians* $G(k, n)$, namely those of type (1, 1, 3) and (1, 2, 2) with $G(1, 4)$, those of type (1, 1, 1, 1, 2) with $G(1, 5)$, those of type (1, 1, 1, 1, 1, 1) with $G(1, 6)$, and those of type (1, 1, 1, 1, 1, 1) with $G(2, 5)$. For these families we have weaker finiteness results, but we will describe conditions sufficient to prove finiteness, and some of the geometry involved. The existence question has been answered, in positive terms for curves with $g = 0$, for the types (1, 1, 3), (1, 2, 2), and (1, 1, 1, 1, 2). See [Kn2].

In Section 4 we will study rational curves in families of Calabi-Yau threefolds F on four-dimensional rational normal scrolls in projective spaces. These threefolds will then correspond to sections of the anti-canonical line bundle on the scroll, a very simple case of “complete intersection”.

The main result of this paper, Theorem 4.3, ensures existence of at least one isolated smooth rational curve of given fixed topological type on general F , for each topological type (a bidegree (d, a)) within a certain range. The main steps of the proof of Theorem 4.3 are sketched in Section 4. There we also briefly investigate the possibilities

for showing finiteness, applying similar methods developed to handle the complete intersection cases described above.

In Sections 5 and 6 are devoted to the details of the proof of Theorem 4.3. In Section 5 we describe some general, useful facts about polarized $K3$ surfaces, and we make some specific lattice-theoretical considerations that will be useful to us.

In Section 6 we prove Theorem 4.3 step by step. In Step (I) we prove Proposition 6.2, which describes curves on $K3$ surfaces, in Steps (II)-(IV) we produce threefolds, which are unions of one-dimensional families of $K3$ surfaces. We produce the desired rational curves on such threefolds and on smooth deformations of such threefolds.

We thank the referee for helpful remarks, and the organizers of the Kleiman's 60th Birthday Conference for making this volume possible. The second author was supported by a grant from the Research Council of Norway.

1.1. Conventions and definitions. The ground field is the field of complex numbers. We say a curve C in a variety V is *geometrically rigid* in V if the space of embedded deformations of C in V is zero-dimensional. If furthermore this space is reduced, we say that C is *infinitesimally rigid* or *isolated* in V .

A *curve* will always be reduced and irreducible.

2. COMPLETE INTERSECTION CALABI-YAU THREEFOLDS IN PROJECTIVE SPACES

In this section we will sketch the situation for the complete intersection Calabi-Yau threefolds in projective spaces. In [Jo-K11] one wrote, regarding the finiteness question for (smooth) rational curves: “The authors have checked the key details, and believe the following ranges come out: $d \leq 7$ for types (3,3) and (2,4), and $d \leq 6$ for types (2,2,3) and (2,2,2,2). In fact, except for the case $d = 6$ and F of type (2,2,2,2), the incidence scheme I_d of pairs (C, F) is, almost certainly irreducible, generically reduced, and of the same dimension as the space \mathbf{P} of F .” Moreover, the “full theorem” for smooth rational curves in quintics, parallel to that of [Nj], was:

Theorem 2.1. *Let $d \leq 9$, and let F be a general quintic threefold in \mathbf{P}^4 . In the Hilbert scheme of F , form the open subscheme of rational, smooth and irreducible curves C of degree d . Then this subscheme is finite, nonempty, and reduced; in fact, each C is embedded in F with normal bundle $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$.*

We are now ready to give corresponding results for the four other complete intersection cases:

Theorem 2.2. *Assume we are in one of the following cases:*

- (a) $d \leq 7$, and F is a general complete intersection threefold of type $(2,4)$ or $(3,3)$ in \mathbf{P}^5 ;
- (b) $d \leq 6$, and F is a general complete intersection threefold of type $(2,2,3)$ in \mathbf{P}^6 ;
- (c) $d \leq 5$, and F is a general complete intersection threefold of type $(2,2,2,2)$ in \mathbf{P}^5 ;

In the Hilbert scheme of F , form the open subscheme of rational, smooth and irreducible curves C of degree d . Then this subscheme is finite, nonempty, and reduced; in fact, each C is embedded in F with normal bundle $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$. Moreover, in the cases $(2,2,3)$ and $d = 7$, and in the case $(2,2,2,2)$ and $d = 6$, this subscheme is finite and non-empty, and there exists a rational curve C in F with normal sheaf $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$.

Proof. In all cases of (a), (b), and (c) one proceeds as follows: Let I_d be the natural incidence of smooth rational curves C and smooth complete intersection threefolds F in question. One then shows that I_d is irreducible, and $\dim I_d = \dim G$, where G is the parameter space of complete intersection threefolds in question. This is enough to prove finiteness. In [Jor] not only the key details, but a complete proof of this result, was given.

Secondly, for each of the 5 intersection types (of CICY's in some \mathbf{P}^n) one has the following existence result, proven in [K12] and [E-J-S] (Oguiso settled the $(2,4)$ case in [Og]) It is also a special case of [Kn2, Thm. 1.1 and Rem. 1.2] and [K11, Thm. 2.1].

Theorem 2.3. *For all natural numbers d there exists a smooth rational curve C of degree d and a smooth CICY F , with normal sheaf $N_{C|F} = \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$ (which gives $h^0(N_{C|F}) = 0$).*

Using these two pieces of information Theorem 2.2 follows as in [Ka, p. 152-153].

In the cases $(2,2,3)$ and $d = 7$, and $(2,2,2,2)$ and $d = 6$, one proves $\dim I_d = \dim G$ and combines with Theorem 2.3.

□

3. CICY THREEFOLDS IN GRASSMANNIANS

There are several ways of describing and compactifying the set of smooth rational curves of degree d in the Grassmann variety $G(k, n)$.

See for example [Str]. Let $M_{d,k,n}$ denote the Hilbert scheme of smooth rational curves of degree d in $G(k, n)$. It is well known that the dimension of $M_{d,k,n}$ is $(n+1)d + (k+1)(n-k) - 3$.

Let each $G(k, n)$ be embedded in \mathbf{P}^N , where $N = \binom{n+1}{k+1} - 1$, by the Plücker embedding. Let G parametrize the set of smooth complete intersection threefolds with $G(k, n)$ by hypersurfaces of degrees (a_1, \dots, a_s) in \mathbf{P}^N , where $s = \dim G(k, n) - 3 = (k+1)(n-k) - 3$, and $a_1 + \dots + a_s = n+1$. Adjunction gives that the complete intersections thus defined have trivial canonical sheaves, and thus are Calabi-Yau threefolds. An easy numerical calculation gives that there are five families of Calabi-Yau threefolds F that are complete intersections with Grassmannians $G(k, n)$, beside those that are straightforward complete intersections of projective spaces \mathbf{P}^N (corresponding to the special case $k=0, n=N$). It will be natural for us to divide these five cases into two categories:

- (a) Those where $a_i = 1$, for all i . These are of type $(1, 1, 1, 1, 1, 1)$ in $G(1, 6)$ in \mathbf{P}^{20} , or of type $(1, 1, 1, 1, 1, 1)$ in $G(2, 5)$ in \mathbf{P}^{19} . The dimensions of the parameter spaces G of F in question, are 98 and 84, respectively.
- (b) Those where $a_i \geq 2$, for some i . These are of type $(1, 1, 3)$ or $(1, 2, 2)$ in $G(1, 4)$ in \mathbf{P}^9 , or of type $(1, 1, 1, 1, 2)$ in $G(1, 5)$ in \mathbf{P}^{14} . The dimensions of the parameter spaces of F in question, are 135, 95 and 109, respectively.

The existence question for rational curves of all degrees has been settled by the second author in [Kn2, Thm. 1.1 and Rem. 1.2)], where it is concluded that for general F of types $(1, 1, 3)$, $(1, 2, 2)$, or $(1, 1, 1, 1, 2)$, and any integer $d > 0$ there exists a smooth rational curve C of degree d in F with $N_{C|F} = \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$ (which gives $h^0(N_{C|F}) = 0$). For the types $(1, 1, 1, 1, 1, 1)$ and $(1, 1, 1, 1, 1, 1)$ we know of no such result.

The finiteness question seems hard to handle in all these cases. Let I_d be the incidence of C and complete intersection F . Denote by a the projection to the parameter space $M_{d,k,n}$ of points $[C]$ representing smooth rational curves C of degree d in $G(k, n)$, and by b the projection to the parameter space of complete intersections of the $G(k, n)$ in question. A natural strategy is to look at each fixed rational curve C and study the fibre $a^{-1}([C])$. If the dimension of all such fibres can be controlled, so can $\dim I_d$. A way to gain partial control is the following: Let M be a subscheme of $M_{d,k,n}$, with $\dim M = m$, and assume that $\dim a^{-1}([C]) = c$ is constant on M . Then of course this gives rise

to a part $a^{-1}(M)$ of I_d which has dimension $c + m$. If $c + m \leq \dim G$, one concludes that for a general point $[g]$ of G there is only a finite set of points from $a^{-1}(M)$ in $b^{-1}([g])$. More refined arguments may reveal that in many such cases $a^{-1}(M)$ is irreducible.

An obvious argument shows that if M is the subscheme of $M(d, k, n)$ corresponding to rational normal curves of degree d , then $\dim a^{-1}([C])$ is constant on M . Moreover the constant value is $c = \dim G - \dim M_{d,k,n} = \dim G - ((n+1)d + (k+1)(n-k) - 3)$. Of course the rational normal curves for fixed n, k only occur for (low) $d \leq N - r$, where $G(k, n)$ is embedded in \mathbf{P}^N , and r is the number of i with $a_i = 1$. (One observes that $N - r = \max\{d \mid \dim G - \dim M_{d,k,n} \geq 0\}$ for the cases in category (a), but $N - r < \max\{d \mid \dim G - \dim M_{d,k,n} \geq 0\}$ for the cases in category (b).)

For $d = 1, 2, 3$ the only smooth rational curves of degree d are the rational, normal ones. In [Os] it was shown for all 5 cases that I_d is irreducible for $d = 1, 2, 3$, and that on a general F there is no singular (plane) cubic curve on F . In a similar way one can show that on a general F there is no pair of intersecting lines, and no line intersecting a conic and no double line. For the complete intersection types of category (b) one then has:

Proposition 3.1. *(i) Let $d \leq 3$, and let F be a general threefold of a given type as described above. In the Hilbert scheme of F , form the open subscheme of rational, smooth and irreducible curves C of degree d . Then this subscheme is finite, nonempty, and reduced; in fact, each C is embedded in F with normal bundle $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$. Moreover there are no singular curves (reducible or irreducible) of degree d in F . We have $\dim I_d = \dim G$, and I_d is irreducible.*

(ii) For all natural numbers d the incidence I_d contains a component of dimension $\dim G$, and this component dominates G by the second projection map b .

Proof. Part (i) follows from the irreducibility of I_d , for $d = 1, 2, 3$, and the existence result in [Kn2], using [Ka, p. 152-153]. Part (ii) follows from the same results, focusing only on one particular component of I_d , corresponding to the curve found in [Kn2]. \square

In [B-C-K-S], one finds virtual numbers of rational curves of degree d on a generic threefold of each of the 5 types described. For the ones of category (b) there should be no problem in interpreting these numbers as actual numbers of smooth rational curves of degree d in a

generic F , for $d = 1, 2, 3$, but it is a challenge to prove the analogue of Part (i) for higher d .

3.1. An analysis of the incidence I_d . For each of the five types one might ask whether it is reasonable to believe that $\dim I_d = \dim G$ (or equivalently: $\dim I_d \leq \dim G$) for many more d , or even for all d . We will point out below that the number of such d is very limited.

We recall that in the well known case of the hyperquintics in \mathbf{P}^4 we have I_d irreducible and of dimension $125 = \dim G$, for $d \leq 9$, reducible for $d = 12$, and reducible with at least one component of dimension at least 126 for $d \geq 13$. The cases $d = 10, 11$ seem to represent “open territory”, while it is also open whether some component has dimension at least 126 for $d = 12$. See [Jo-K12]. (None of these pieces of information contradict Clemens conjecture, which predicts that all components of dimension at least 126 project to some subset of G of positive codimension).

In each of the five types of complete intersection with Grassmannians $G(k, n)$, a similar phenomenon occurs. The most transparent example is perhaps that of threefolds of intersection type (1^7) of $G(1, 6)$ in \mathbf{P}^{20} . We now will show that in this case $\dim I_d > \dim G$ for all $d \geq 4$:

For all points P of \mathbf{P}^6 , look at $H_P = \mathbf{P}^5$ in $G(1, 6)$ parametrizing all lines through P . The subset of $M_{d,k,n}$ parametrizing curves in H_P , only spanning a \mathbf{P}^3 (inside H_P inside $G(1, 6)$ inside \mathbf{P}^{20}) has dimension $4d + 8$. There is a 70-dimensional family of 13-planes in \mathbf{P}^{20} containing a given \mathbf{P}^3 . Hence $\dim I_d \geq 6 + (4d + 8) + 70 = 4d + 84$. Clearly this exceeds 98 for $d \geq 4$.

Let J be the subset of I_d thus obtained. The set of 3-spaces contained in some H_P has dimension $6 + \dim G(3, 5) = 14$, so $\dim b(J) \leq 14 + 70 = 84 < 98$. Hence J , although big, gives no contradiction to the analogue of Clemens conjecture.

To complete the picture we will also exhibit another part of the incidence I_d of dimension $4d + 69$. This is larger than 98 for $d \geq 8$. We will study the part of I_d that arises from curves C in $G(1, 6)$, such that its associated ruled surface in \mathbf{P}^6 only spans a \mathbf{P}^3 inside that \mathbf{P}^6 . Each such curve is contained in a $G(1, 3)$ in a \mathbf{P}^5 inside \mathbf{P}^{20} , and a simple dimension count gives dimension $4d + 69$.

On the other hand, it is for example clear that a general \mathbf{P}^{13} inside \mathbf{P}^{20} (corresponding to a general (1^7) of $G(1, 6)$) does not contain a \mathbf{P}^5 spanned by a sub- $G(1, 3)$ of $G(1, 6)$. This means that the subsets of I_d , corresponding to curves C contained in a $G(1, 3)$, such that C

and $G(1, 3)$ span the same \mathbf{P}^5 , project by b to subsets of G of positive codimension. Some Schubert calculus reveals that the same is true for the part of I_d corresponding to those C contained in a $G(1, 3)$ and spanning at most a \mathbf{P}^4 also. Hence the “problematic” part of I_d in consideration here does not give a contradiction to the analogue of Clemens conjecture.

For $d \geq 12$ the part of I_d arising from curves such that its associated ruled surface spans a \mathbf{P}^4 inside \mathbf{P}^6 , will have dimension at least $5d + 41$, which is larger than 98, for $d \geq 12$. As above we see that general \mathbf{P}^{13} inside \mathbf{P}^{20} does not contain a \mathbf{P}^9 spanned by a sub- $G(1, 4)$ of $G(1, 6)$.

For $d \geq 15$ the part of I_d arising from curves such that their associated ruled surfaces spans a \mathbf{P}^5 inside \mathbf{P}^6 , will have dimension at least 99.

Let J be the subset of I_d thus obtained. The set of 3-spaces contained in some H_P has dimension $6 + \dim G(3, 5) = 14$, so $\dim b(J) \leq 14 + 70 = 84 < 98$. Hence J gives no contradiction to the analogue of Clemens’ conjecture.

A similar phenomenon occurs for the case of threefolds of intersection type (1^6) of $G(2, 5)$ in \mathbf{P}^{19} . We recall $\dim G = 84$ in this case. For a given C in $G(2, 5)$ the associated ruled threefold in \mathbf{P}^5 may span a 3-space, a 4-space, or all of \mathbf{P}^5 . The former ones give rise to a part of the incidence of dimension $4d + 68$. This is equal to $\dim G$ for $d = 4$, and I_d is reducible then. For $d \geq 5$ we see that $\dim I_d > \dim G$.

4. RATIONAL CURVES IN SOME CY THREEFOLDS IN FOUR-DIMENSIONAL RATIONAL NORMAL SCROLLS

In this section we state the main result of this paper, Theorem 4.3, and we sketch the main steps of its proof. We also give some supplementary results, and remark on the possibility of finding analogues of our main result.

We start by reviewing some basic facts about rational normal scrolls.

Definition 4.1. Let $\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_d)$, with $e_1 \geq \cdots \geq e_d \geq 0$ and $f = e_1 + \cdots + e_d \geq 2$. Consider the line bundle $\mathcal{L} = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ on the corresponding \mathbf{P}^{d-1} -bundle $\mathbf{P}(\mathcal{E})$ over \mathbf{P}^1 . We map $\mathbf{P}(\mathcal{E})$ into \mathbf{P}^N with the complete linear system $|\mathcal{L}|$, where $N = f + d - 1$. The image \mathcal{T} is by definition a rational normal scroll of type (e_1, \dots, e_d) . The image is smooth, and isomorphic to $\mathbf{P}(\mathcal{E})$, if and only if $e_d \geq 1$.

Definition 4.2. Let \mathcal{T} be a rational normal scroll of type (e_1, \dots, e_d) . We say that \mathcal{T} is a scroll of maximally balanced type if $e_1 - e_d \leq 1$.

Denote by \mathcal{H} the hyperplane section of a rational normal scroll \mathcal{T} , and let C be a (rational) curve in \mathcal{T} . We say that the *bidegree* of C is (d, a) if $\deg C = C \cdot \mathcal{H} = d$, considered as a curve on projective space, and $C \cdot \mathcal{F} = a$, where \mathcal{F} is the fiber of the scroll.

From now on we will let \mathcal{T} be a rational normal scroll of dimension 4 in \mathbf{P}^N , and of type (e_1, \dots, e_4) , where the e_i are ordered in a non-increasing way, and $e_1 - e_3 \leq 1$. Hence the subscroll $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \mathcal{O}_{\mathbf{P}^1}(e_2) \oplus \mathcal{O}_{\mathbf{P}^1}(e_3))$ is of maximally balanced type. Moreover we are especially interested in the case where the general 3-dimensional (anti-canonical) divisor of type $4\mathcal{H} - (N - 5)\mathcal{F}$ is non-singular and thus a Calabi-Yau threefold. Then we have to restrict to 5 families of subcases. We will show that for positive a , and d exceeding a lower bound, depending on a , a general 3 dimensional (anti-canonical) divisor of type $4\mathcal{H} - (N - 5)\mathcal{F}$ will contain an isolated rational curve of bidegree (d, a) . To be more precise, we will show:

Theorem 4.3. *Let \mathcal{T} be a rational normal scroll of dimension 4 in \mathbf{P}^N with a balanced subscroll of dimension 3 as described. Assume this subscroll spans a \mathbf{P}^g (so $g = e_1 + e_2 + e_3 + 2$) Let $d \geq 1$, and $a \geq 1$, be integers satisfying the following conditions:*

- (i) *If $g \equiv 1 \pmod{3}$, then either $(d, a) \in \{(\frac{g-1}{3}, 1), (2(g-1)/3, 2)\}$; or $d > \frac{(g-1)a}{3} - \frac{3}{a}$, $(d, a) \neq (2(g-1)/3 - 1, 2)$ and $3d \neq (g-1)a$.*
- (ii) *If $g \equiv 2 \pmod{3}$, then either $(d, a) \in \{(g-1, 3), (2g-2, 6)\}$; or $d > \frac{(g-1)a}{3} - \frac{3}{a}$, $(d, a) \notin \{(2(g-2)/3, 2), ((4g-5)/3, 4), ((7g-8)/3, 7)\}$ and $3d \neq (g-1)a$.*
- (iii) *If $g \equiv 0 \pmod{3}$, then either $(d, a) \in \{((g-3)/3, 1), ((2g-3)/3, 2)\}$; or $d \geq ga/3$.*

Then the zero scheme of a general section of $4\mathcal{H} - (N - 5)\mathcal{F}$ will be a (possibly singular) Calabi-Yau threefold X that contains an isolated rational curve of bidegree (d, a) lying outside of the singular locus of X . If moreover the scroll type (e_1, \dots, e_4) is of one of the following forms: (s, s, s, s) , $(s+1, s, s, s)$, $(s+1, s+1, s, s)$, $(s+1, s+1, s+1, s)$, $(s+2, s+1, s+1, s)$, for $s \geq 1$, then the zero scheme of a general section of $4\mathcal{H} - (N - 5)\mathcal{F}$ will in addition be a smooth Calabi-Yau threefold.

The theorem will be proved in several steps. The fact that a general section is smooth in the 5 subcases follows directly from Bertini's theorem if $4\mathcal{H} - (N - 5)\mathcal{F}$ is base point free. The divisor is anti-canonical and of dimension 3. Hence it will be a Calabi-Yau threefold V if $h^1(\mathcal{O}_V) = h^2(\mathcal{O}_V) = 0$. These numbers are zero, because of the short

exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{T}}(K_{\mathcal{T}}) \rightarrow \mathcal{O}_{\mathcal{T}} \rightarrow \mathcal{O}_V \rightarrow 0,$$

which gives rise to the cohomology sequence

$$\begin{aligned} H^1(\mathcal{O}_{\mathcal{T}}) &\rightarrow H^1(\mathcal{O}_V) \rightarrow \\ H^2(\mathcal{O}_{\mathcal{T}}(K_{\mathcal{T}})) &\rightarrow H^2(\mathcal{O}_{\mathcal{T}}) \rightarrow H^2(\mathcal{O}_V) \rightarrow H^3(\mathcal{O}_{\mathcal{T}}(K_{\mathcal{T}})). \end{aligned}$$

Since $h^i(\mathcal{O}_{\mathcal{T}}(K_{\mathcal{T}})) = h^{4-i}(\mathcal{O}_{\mathcal{T}}) = 0$, for $i = 1, 2, 3$, for example by Remark 1.4 of [SD], we conclude that $h^1(\mathcal{O}_V) = h^2(\mathcal{O}_V) = 0$

If the scroll type is (s, s, s, s) , $(s+1, s, s, s)$ or $(s+1, s+1, s, s)$, then $4\mathcal{H} - (N-5)\mathcal{F}$ is base point free (as it is in a fourth case $(s+2, s, s, s)$, which is a case not considered in our result, since the subscroll $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \mathcal{O}_{\mathbf{P}^1}(e_2) \oplus \mathcal{O}_{\mathbf{P}^1}(e_3))$ is not maximally balanced then). This follows for example from using coordinates on rational normal scrolls, as described in [Sc], p. 110-11, or in [Ste]. Using the same kind of coordinates, one finds that in the cases $(s+1, s+1, s+1, s)$ or $(s+2, s+1, s+1, s)$ the base locus is the fourth directrix curve, and that by a properly generalized Bertini's theorem the general section of $4\mathcal{H} - (N-5)\mathcal{F}$ is non-singular outside this directrix curve. Moreover a more refined study reveals that the general section of $4\mathcal{H} - (N-5)\mathcal{F}$ is smooth at all points of this base locus simultaneously. Hence the general section is a Calabi-Yau threefold also in these two cases. Here are the main steps in the proof of the statement about the existence of an isolated curve as described:

- (I) Set $g := e_1 + e_2 + e_3 + 2$. Using lattice-theoretical considerations we find a (smooth) $K3$ surface S in \mathbf{P}^g with $\text{Pic } S \simeq \mathbf{Z}H \oplus \mathbf{Z}D \oplus \mathbf{Z}\Gamma$, where H is the hyperplane section class, D is the class of a smooth elliptic curve of degree 3 and Γ is a smooth rational curve of bidegree (d, a) . Let $T = T_S$ be the 3-dimensional scroll in \mathbf{P}^g swept out by the linear spans of the divisors in $|D|$ on S . The rational normal scroll T will be of maximally balanced type and of degree $e_1 + e_2 + e_3$.
- (II) Embed $T = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \mathcal{O}_{\mathbf{P}^1}(e_2) \oplus \mathcal{O}_{\mathbf{P}^1}(e_3))$ (in the obvious way) in a 4 dimensional scroll $\mathcal{T} = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_4))$ of type (e_1, \dots, e_4) . Hence T corresponds to the divisor class $\mathcal{H} - e_4F$ in \mathcal{T} , and S corresponds to a “complete intersection” of divisors of type $\mathcal{H} - e_4\mathcal{F}$ and $3\mathcal{H} - (g-4)\mathcal{F}$ on \mathcal{T} . We now deform the complete intersection in a rational family (i.e. parametrized by \mathbf{P}^1) in a general way. For “small values” of the parameter we obtain a $K3$ surface with Picard group of rank 2 and no rational curve on it.

- (III) Take the union over \mathbf{P}^1 of all the K3 surfaces described in (II). This gives a threefold V , which is a section of the anti-canonical divisor $4\mathcal{H} - (g - 4 + e_4)\mathcal{F} = 4\mathcal{H} - (N - 5)\mathcal{F}$ on \mathcal{T} . For a general complete intersection deformation the threefold will have only finitely many singularities outside the fourth directrix curve, and it will be non-singular along Γ . In the five special families it will be smooth also along the fourth directrix curve. Then Γ will be isolated on V .
- (IV) Deform V as a section of $4\mathcal{H} - (g - 4 + e_4)\mathcal{F} = 4\mathcal{H} - (N - 5)\mathcal{F}$ on \mathcal{T} . Then a general deformation W will have an isolated curve Γ_W of bidegree (d, a) . In the five special families it will be smooth.

This strategy is analogous to the one used in [Cl1] to show the existence of isolated rational curves of infinitely many degrees in the generic quintic in \mathbf{P}^4 , and in [E-J-S] to show the existence of isolated rational curves of bidegree $(d, 0)$ in general complete intersection Calabi-Yau threefolds in some specific biprojective spaces.

Step (I) will be proved in Sections 5 and 6, and Steps (II)-(IV) in Section 6.

4.1. Finiteness questions. Let us say a few words about finiteness. Let \mathcal{T} be a rational, normal scroll of dimension 4 in \mathbf{P}^N . A divisor of type $4\mathcal{H} - (N - 5)\mathcal{F}$ corresponds to a quartic hypersurface Q containing $N - 5$ given 3-spaces in the \mathcal{F} -fibration of \mathcal{T} . Then, for general such Q , we see that $Q \cap \mathcal{T}$ is the union of a Calabi-Yau threefold and the $N - 5$ given 3-spaces. Each rational curve C of type (d, a) for $a \geq 1$ intersects each 3-space in at most a , and hence a finite number of points.

Let $M = M_{d,a} = \{[C] \mid C \text{ has bidegree } (d, a)\}$, and let G be the parameter space of “hypersurfaces” of type $4\mathcal{H} - (N - 5)\mathcal{F}$. Study the incidence $I = I_{d,a} = \{([C], [F]) \in M \times G \mid C \subset F\}$, and let π be the projection onto the first factor. For a given rational curve C_0 we want to study the following subset

$$\pi_1^{-1}([C_0]) = \{([C_0], [F]) \mid C_0 \in F\}$$

of the incidence $I = I_{d,a}$.

Finding the dimension of $\pi_1^{-1}([C_0])$ is essentially, as we shall see below, equivalent to finding $h^0(\mathcal{J}(4))$ (or $(h^1(\mathcal{J}(4)))$, where \mathcal{J} is the ideal sheaf in \mathbf{P}^N of the union X of C_0 and the $N - 5$ disjoint, linear 3-spaces, each intersecting C_0 as described. We then have the following result, which is Corollary 1.9 of [Si]:

Lemma 4.4. *Let \mathcal{J} be the ideal sheaf of a projective scheme X that consists of the union of d schemes X_1, \dots, X_d in \mathbf{P}^N , whose pairwise intersections are finite sets of points. Let m_i be the regularity of X_i . Then \mathcal{J} is $\sum_{i=1}^d m_i$ -regular.*

We will use this result. Look at the following exact sequence:

$$0 \rightarrow \mathcal{J}_{X/T}(4\mathcal{H}) \rightarrow \mathcal{O}_T(4\mathcal{H}) \rightarrow \mathcal{O}_X(4\mathcal{H}) \rightarrow 0.$$

This gives rise to the exact cohomology sequence

$$0 \rightarrow H^0(\mathcal{J}_{X/T}(4\mathcal{H})) \rightarrow H^0(\mathcal{O}_T(4\mathcal{H})) \rightarrow H^0(\mathcal{O}_X(4\mathcal{H})) \rightarrow H^1(\mathcal{J}_{X/T}(4\mathcal{H})) \rightarrow 0.$$

This gives:

$$\begin{aligned} h^0(\mathcal{J}_{X/T}(4\mathcal{H})) &= h^1(\mathcal{J}_{X/T}(4\mathcal{H})) + h^0(\mathcal{O}_T(4\mathcal{H})) - h^0(\mathcal{O}_X(4\mathcal{H})) = \\ &= h^1(\mathcal{J}_{X/T}(4\mathcal{H})) + 35(N-2) - (35(N-5) + 4d + 1 - (N-5)a) \\ &= h^1(\mathcal{J}_{X/T}(4\mathcal{H})) + 105 - (4d + 1 + (5-N)a). \end{aligned}$$

Hence we see that

$$\begin{aligned} \dim \pi^{-1}([C_0]) &= h^1(\mathcal{J}_{X/T}(4\mathcal{H})) + 104 - \dim M_{d,a} = \\ &= h^1(\mathcal{J}_{X/T}(4\mathcal{H})) + \dim G - \dim M_{d,a}, \end{aligned}$$

if $4e_4 - (N-5) \geq -2$. If we work with a stratum W of $M = M_{d,a}$ where $h^1(\mathcal{J}_{X/T}(4\mathcal{H}))$ is constant, say c , then the incidence stratum $\pi^{-1}(W)$ has dimension $c + \dim G - \text{codim}(W, M)$. Now it is clear that $h^1(\mathcal{J}_{X/T}(4\mathcal{H})) = h^1(\mathcal{J}_{X/\mathbf{P}^N}(4)) = h^1(\mathcal{J}(4))$, since $h^1(\mathcal{J}_{T/\mathbf{P}^N}(4)) = 0$, which is true because rational normal scrolls are projectively normal. Moreover $h^1(\mathcal{J}(4)) = 0$ if X is 5-regular, by the definition of m -regularity in general. By Theorem 1.1 of [G-L-P] we have:

Lemma 4.5. *A non-degenerate, reduced, irreducible curve of degree d in \mathbf{P}^r is $(d+2-r)$ -regular.*

Moreover in Corollary 1.10 of [Si] one has:

Lemma 4.6. *The ideal sheaf of s linear k -spaces meeting (pairwise) in finitely many (or no) points is s -regular.*

Putting these two results together, we observe that if C_0 spans an r -space, then X is $(d+2-r+N-5) = (d-r+N-3)$ -regular. In particular, if C_0 is a rational normal curve, then X is $(N-3)$ -regular, and in particular 5-regular if N is 7 or 8 (and of course $d \leq N$ then). Also curves spanning a $(d-1)$ -space are 5-regular if $N = 7$ (for $d \leq 8$). We then have:

Corollary 4.7. *On a general F in \mathcal{T} of type $(1, 1, 1, 1)$ in \mathbf{P}^7 there are only finitely many smooth rational curves of degree at most 4. On a general F in \mathcal{T} of scroll type $(2, 1, 1, 1)$ in \mathbf{P}^8 there are only finitely many smooth rational curves of degree at most 3.*

Proof. We deduce that all X in question are 5-regular, so $h^1(\mathcal{J}_{X/\mathbf{P}^N}(4\mathcal{H})) = 0$ for all X , and hence all non-empty incidence varieties $I_{d,a}$ have dimension equal to $\dim G$, and hence the second projection map π_2 has finite fibres over general points of G . \square

4.2. Analogous questions for other threefolds. We would like to remark on the possibility of finding an analogue of Theorem 4.3. Is it possible to produce isolated, rational curves of bidegree (d, a) for many (d, a) , also on general CY threefolds of intersection type $(2\mathcal{H} - c_1\mathcal{F}, 3\mathcal{H} - c_2\mathcal{F})$ on five-dimensional rational normal scrolls in \mathbf{P}^N ? Here we obviously look at fixed (c_1, c_2) such that $c_1 + c_2 = N - 6$.

A natural strategy, analogous to that in the previous section, would be to limit oneself to work with rational normal scrolls $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_5))$ such that $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_4))$ is maximally balanced given its degree (that is: $e_1 - e_4 \leq 1$).

A natural analogue to Step (I) in the proof of Theorem 4.3 in the previous section is: Set $g = e_1 + \cdots + e_4 + 3$. Using lattice-theoretical considerations again, and with certain conditions on n , d and a , we find a (smooth) $K3$ surface S in \mathbf{P}^g with $\text{Pic } S \simeq \mathbf{Z}H \oplus \mathbf{Z}D \oplus \mathbf{Z}\Gamma$, where H is the hyperplane section class, D is the class of a smooth elliptic curve of degree 4 and C is a smooth rational curve of bidegree (d, a) . In particular, S has Clifford index 2 (see Section 5.1 for the definition of the Clifford index of a $K3$ surface). Let $T = T_S$ be the 4-dimensional scroll in \mathbf{P}^g swept out by the linear spans of the divisors in $|D|$ on S . The rational normal scroll T will be maximally balanced of degree $e_1 + e_2 + e_3 + e_4$. In other words we should find an analogue of Proposition 6.2. It seems clear that we can do this.

The natural analogue of Step (II) in the previous section is:

Embed $T = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_4))$ (in the obvious way) in a 5 dimensional scroll $\mathcal{T} = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_5))$ of type (e_1, \dots, e_5) . Hence T corresponds to the divisor class $\mathcal{H} - e_5\mathcal{F}$ in \mathcal{T} .

If g is odd, one would like to show that S is a “complete intersection” of 2 divisors, both of type $2\mathcal{H} - \frac{g-5}{2}\mathcal{F}$ restricted to T . Therefore it is a “complete intersection” of 3 divisors Z_5, Q_1, Q_2 , the first of type $\mathcal{H} - e_5\mathcal{F}$, and the two Q_i of type $2\mathcal{H} - \frac{g-5}{2}\mathcal{F}$ on \mathcal{T} . At the moment we have no water-proof argument for this.

If g is even, one can show that S is a “complete intersection” of 2 divisors, of type $2\mathcal{H} - \frac{g-4}{2}\mathcal{F}$ and $2\mathcal{H} - \frac{g-6}{2}\mathcal{F}$, restricted to T . Therefore it is a “complete intersection” of 3 divisors Z_5, Q_1, Q_2 , of types $\mathcal{H} - e_5\mathcal{F}$, $2\mathcal{H} - \frac{g-4}{2}\mathcal{F}$, and $2\mathcal{H} - \frac{g-6}{2}\mathcal{F}$ on \mathcal{T} .

If one really obtains a complete intersection as described above, one might deform it in a rational family (i.e. parametrized by \mathbf{P}^1). If S is given by equations: $Q_1 = Q_2 = 0$, one looks at deformations:

$$Q_1 + sQ'_1 = Q_2 = Z_5 - sB(t, u, Z_1, \dots, Z_5) = 0,$$

or:

$$Q_1 = Q_2 + sQ'_2 = Z_5 - sB(t, u, Z_1, \dots, Z_5) = 0.$$

Here the B correspond to sections of $\mathcal{H} - e_5\mathcal{F}$. For “small values” of the parameter one would like to obtain a $K3$ surface with Picard group of rank 2, Clifford index 1, (see Section 5.1 for the definition of the Clifford index of a $K3$ surface) and no rational curve on it. For g odd, there is no essential difference between the two types of deformations. For g even, the two deformations are different.

An analogue of Step (III) is: Eliminating s from the first set of equations, we obtain:

$$Q_1B + Z_5Q'_1 = Q_2 = 0.$$

Eliminating s from the second set of equations, we obtain:

$$Q_1 = Q_2B + Z_5Q'_2 = 0.$$

If g is odd, we obtain in both cases a “complete intersection” threefold of type

$$(2\mathcal{H} - \frac{g-5}{2}\mathcal{F}, 3\mathcal{H} - (\frac{g-5}{2} + e_5)\mathcal{F}).$$

If g is even, the first threefold is of type

$$(2\mathcal{H} - \frac{g-6}{2}\mathcal{F}, 3\mathcal{H} - (\frac{g-4}{2} + e_5)\mathcal{F}),$$

while the second is of type

$$(2\mathcal{H} - \frac{g-4}{2}\mathcal{F}, 3\mathcal{H} - (\frac{g-6}{2} + e_5)\mathcal{F}),$$

Since $g = N - 1 - e_5$, we see that in all cases we have intersection type $(2\mathcal{H} - c_1\mathcal{F}, 3\mathcal{H} - c_2\mathcal{H})$, such that $c_1 + c_2 = N - 6$.

The analogue of Step (IV) seems doable for g odd, but here Step (II), as remarked, is unclear. The details of this analogue for g even are also not quite clear to us.

5. K3 SURFACE COMPUTATIONS

The purpose of the section is to make the necessary technical preparations to complete Step (I) of the proof of Theorem 4.3. First we will recall some useful facts about $K3$ surfaces and rational normal scrolls. In Lemma 5.3 we introduce a specific $K3$ surface which will be essential in the proof of Step (I). In the last part of the section we make some $K3$ theoretical computations related to the Picard lattice of this particular $K3$ surface.

5.1. Some general facts about $K3$ surfaces. Recall that a $K3$ surface is a (reduced and irreducible) surface S with trivial canonical bundle and such that $H^1(\mathcal{O}_S) = 0$. In particular $h^2(\mathcal{O}_S) = 1$ and $\chi(\mathcal{O}_S) = 2$.

We will use line bundles and divisors on a $K3$ surface with little or no distinction, as well as the multiplicative and additive notation, and denote linear equivalence of divisors by \sim .

Before continuing, we briefly recall some useful facts and some of the main results in [Jo-Kn] which will be used in the proof of Theorem 4.3.

Let C be a smooth irreducible curve of genus $g \geq 2$ and A a line bundle on C . The *Clifford index* of A (introduced by H. H. Martens in [Ma]) is the integer

$$\text{Cliff } A = \deg A - 2(h^0(A) - 1).$$

If $g \geq 4$, then the *Clifford index of C* itself is defined as

$$\text{Cliff } C = \min\{\text{Cliff } A \mid h^0(A) \geq 2, h^1(A) \geq 2\}.$$

Clifford's theorem then states that $\text{Cliff } C \geq 0$ with equality if and only if C is hyperelliptic and $\text{Cliff } C = 1$ if and only if C is trigonal or a smooth plane quintic.

At the other extreme, we get from Brill-Noether theory (cf. [A-C-G-H], Chapter V) that $\text{Cliff } C \leq \lfloor \frac{g-1}{2} \rfloor$. For the general curve of genus g , we have $\text{Cliff } C = \lfloor \frac{g-1}{2} \rfloor$.

We say that a line bundle A on C *contributes to the Clifford index of C* if $h^0(A), h^1(A) \geq 2$ and that it *computes the Clifford index of C* if in addition $\text{Cliff } C = \text{Cliff } A$.

Note that $\text{Cliff } A = \text{Cliff } \omega_C \otimes A^{-1}$.

It was shown by Green and Lazarsfeld [Gr-La] that the Clifford index is constant for all smooth curves in a complete linear system $|L|$ on a $K3$ surface. Moreover, they also showed that if $\text{Cliff } C < \lfloor \frac{g-1}{2} \rfloor$ (where g denotes the sectional genus of L , i.e. $L^2 = 2g - 2$), then there

exists a line bundle M on S such that $M_C := M \otimes \mathcal{O}_C$ computes the Clifford index of C for all smooth irreducible $C \in |L|$.

This was investigated further in [Jo-Kn], where we defined the Clifford index of a base point free line bundle L on a $K3$ surface to be the Clifford index of all the smooth curves in $|L|$ and denoted it by $\text{Cliff } L$. Similarly, if (S, L) is a polarized $K3$ surface we defined the Clifford index of S , denoted by $\text{Cliff}_L(S)$ to be $\text{Cliff } L$.

The following is a summary of the results obtained in [Jo-Kn], that we will need in the following. Since we only need those results for ample L , we restrict to this case and refer the reader to [Jo-Kn] for the results when L is only assumed to be base point free.

Proposition 5.1. *Let L be an ample line bundle of sectional genus $g \geq 4$ on a $K3$ surface S and let $c := \text{Cliff } L$. Assume $c < \lfloor \frac{g-1}{2} \rfloor$. Then c is equal to the minimal integer $k \geq 0$ such that there is a line bundle D on S satisfying the numerical conditions:*

$$2D^2 \stackrel{(i)}{\leq} L.D = D^2 + k + 2 \stackrel{(ii)}{\leq} 2k + 4$$

with equality in (i) or (ii) if and only if $L \sim 2D$ and $L^2 = 4k + 8$.

(In particular,

$$D^2 \leq c + 2, \text{ with equality if and only if } L \sim 2D \text{ and } L^2 = 4c + 8,$$

and by the Hodge index theorem

$$D^2 L^2 \leq (L.D)^2 = (D^2 + c + 2)^2.)$$

Moreover, any such D satisfies (with $M := L - D$ and $R := L - 2D$):

- (i) $D.M = c + 2$,
- (ii) $D.L \leq M.L$ (equivalently $D^2 \leq M^2$),
- (iii) $h^1(D) = h^1(M) = 0$
- (iv) $|D|$ and $|M|$ are base point free and their generic members are smooth curves,
- (v) $h^1(R) = 0$, $R^2 \geq -4$, and $h^0(R) > 0$ if and only if $R^2 \geq -2$,
- (vi) If $R \sim R_1 + R_2$ is a nontrivial effective decomposition, then $R_1.R_2 > 0$.

Proof. The first statement is [Kn1, Lemma 8.3]. The properties (i)-(iv) are the properties (C1)-(C5) in [Jo-Kn, p. 9-10], under the additional condition that L is ample. The fact that $h^1(R) = 0$ in (v) follows from [Jo-Kn, Prop. 5.5] (where $\Delta = 0$ since L is ample), and the rest of (v) is then an immediate consequence of Riemann-Roch. Finally, (vi) follows from [Jo-Kn, Prop. 6.6] since L is ample. \square

Now denote by ϕ_L the morphism

$$\phi_L : S \longrightarrow \mathbf{P}^g$$

defined by $|L|$ and pick a subpencil $\{D_\lambda\} \subseteq |D| \simeq \mathbf{P}^{\frac{1}{2}D^2+1}$ generated by two smooth curves (so that in particular $\{D_\lambda\}$ is without fixed components, and with exactly D^2 base points). Each $\phi_L(D_\lambda)$ will span a $(h^0(L) - h^0(L - D) - 1)$ -dimensional subspace of \mathbf{P}^g , which is called the linear span of $\phi_L(D_\lambda)$ and denoted by $\overline{D_\lambda}$. Note that $\overline{D_\lambda} = \mathbf{P}^{c+1+\frac{1}{2}D^2}$. The variety swept out by these linear spaces,

$$T = \bigcup_{\lambda \in \mathbf{P}^1} \overline{D_\lambda} \subseteq \mathbf{P}^g,$$

is a rational normal scroll (see [Sc]) of type (e_1, \dots, e_d) , where

$$(1) \quad e_i = \#\{j \mid d_j \geq i\} - 1,$$

with

$$\begin{aligned} d = d_0 &:= h^0(L) - h^0(L - D), \\ d_1 &:= h^0(L - D) - h^0(L - 2D), \\ &\vdots \\ d_i &:= h^0(L - iD) - h^0(L - (i+1)D), \\ &\vdots \end{aligned}$$

Furthermore, T has dimension $\dim T = d_0 = h^0(L) - h^0(F) = c + 2 + \frac{1}{2}D^2$ and degree $\deg T = h^0(F) = g - c - 1 - \frac{1}{2}D^2$.

We will need the following

Lemma 5.2. *Assume L is ample, $D^2 = 0$ and that $h^1(L - iD) = 0$ for all $i \geq 0$ such that $L - iD \geq 0$. Then the scroll T defined by $|D|$ as described above is smooth and of maximally balanced scroll type. Furthermore, $\dim T = c + 2$ and $\deg T = g - c - 1$.*

Proof. Let $r := \max\{i \mid L - iD \geq 0\}$. Then by Riemann-Roch, and our hypothesis that $h^1(L - iD) = 0$ for all $i \geq 0$ such that $L - iD \geq 0$, one easily finds $r = \lfloor \frac{g}{c+2} \rfloor$ and

$$\begin{aligned} d_0 &= \dots = d_{r-1} = L.D = c + 2, \\ 1 \leq d_r &= g + 1 - (c + 2)r \leq c + 2, \\ d_i &= 0 \quad \text{for } i \geq r + 1, \end{aligned}$$

whence the scroll T is smooth and of maximally balanced scroll type. The assertions about its dimension and degree are immediate. \square

5.2. Some specific $K3$ surface computations. In the following lemma we introduce a specific $K3$ surface with a specific Picard lattice, which will be instrumental in proving Theorem 4.3. The element Γ in the lattice will correspond to a curve of bidegree (d, a) as described in that theorem.

Lemma 5.3. *Let $n \geq 4$, $d > 0$ and $a > 0$ be integers satisfying $d > \frac{na}{3} - \frac{3}{a}$. Then there exists an algebraic $K3$ surface S with Picard group $\text{Pic } S \simeq \mathbf{Z}H \oplus \mathbf{Z}D \oplus \mathbf{Z}\Gamma$ with the following intersection matrix:*

$$\begin{bmatrix} H^2 & H.D & H.\Gamma \\ D.H & D^2 & D.\Gamma \\ \Gamma.H & \Gamma.D & \Gamma^2 \end{bmatrix} = \begin{bmatrix} 2n & 3 & d \\ 3 & 0 & a \\ d & a & -2 \end{bmatrix}$$

and such that the line bundle $L := H - \lfloor \frac{n-4}{3} \rfloor D$ is nef.

Proof. The signature of the matrix above is $(1, 2)$ under the given conditions. By a result of Nikulin [Ni] (see also [Morr, Theorem 2.9(i)]) there exists an algebraic $K3$ surface S with Picard group $\text{Pic } S = \mathbf{Z}H \oplus \mathbf{Z}D \oplus \mathbf{Z}\Gamma$ and intersection matrix as indicated.

Since $L^2 > 0$, we can, by using Picard-Lefschetz transformations, assume that L is nef (see e.g. [Og] or [Kn3]). \square

Note now that

$$(2) \quad L^2 = \begin{cases} 8 & \text{if } n \equiv 4 \pmod{3}, \\ 10 & \text{if } n \equiv 5 \pmod{3}, \\ 12 & \text{if } n \equiv 6 \pmod{3}. \end{cases}$$

We will from now on write $L^2 = 2m$, for

$$(3) \quad m := n - 3\lfloor \frac{n-4}{3} \rfloor = 4, 5 \text{ or } 6$$

(in other words $n \equiv m \pmod{3}$) and define

$$(4) \quad d_0 := \Gamma.L = d - \lfloor \frac{n-4}{3} \rfloor a > 0.$$

Note that the condition $d > \frac{na}{3} - \frac{3}{a}$ is equivalent to

$$(5) \quad d_0 > \frac{ma}{3} - \frac{3}{a}.$$

Also note that $\text{Pic } S \simeq \mathbf{Z}L \oplus \mathbf{Z}D \oplus \mathbf{Z}\Gamma$, and that

(6)

$$\delta := |\text{disc}(L, D, \Gamma)| = |\text{disc}(H, D, \Gamma)| = |2a(3d-na)+18| = |2a(3d_0-ma)+18|$$

divides $|\text{disc}(A, B, C)|$ for any $A, B, C \in \text{Pic } S$.

Now we will study the $K3$ surface defined in Lemma 5.3 in further detail.

Lemma 5.4. *Let S, H, D, Γ and L be as in Lemma 5.3. Then L is base point free and $\text{Cliff } L = 1$. Furthermore, L is ample (whence very ample) if and only if we are not in any of the following cases:*

- (a) $ma = 3d_0$ and $9 \mid ma$,
- (b) $m = 4$ and $(d_0, a) = (2, 2), (5, 4)$ or $(9, 7)$,
- (c) $m = 5$ and $(d_0, a) = (2, 2), (6, 4)$ or $(13, 8)$,
- (d) $m = 6$ and $(d_0, a) = (3, 2)$.

Moreover, if L is ample, then $|D|$ is a base point free pencil, $L - D$ is also base point free and $h^1(L - D) = h^1(L - 2D) = 0$.

Proof. Since $D.L = 3$, we have $\text{Cliff } L \leq 1$ by Proposition 5.1. To prove the two first statements, it suffices to show (by classical results on line bundles on $K3$ surfaces such as in [SD] and by Proposition 5.1) that there is no smooth curve E satisfying $E^2 = 0$ and $E.L = 1, 2$.

Since E is base point free, being a smooth curve of non-negative self-intersection (see [SD]), we must have $E.D \geq 0$. If $E.D = 0$, then the divisor $B := 3E - (E.L)D$ satisfies $B^2 = 0$ and $B.L = 0$, whence by the Hodge index theorem we have $3E \sim (E.L)D$, contradicting that D is part of a basis of $\text{Pic } S$. If $E.D \geq 2$, the Hodge index theorem gives the contradiction

$$32 \leq 2L^2(E.D) = L^2(E + D)^2 \leq (L.(E + D))^2 \leq 25.$$

We now treat the case $E.D = 1$. If $E.L = 1$, we get

$$16 \leq 2L^2 = L^2(E + D)^2 \leq (L.(E + D))^2 = 16,$$

whence by the Hodge index theorem $L \sim 2(E + D)$, which is impossible, since L is part of a basis of $\text{Pic } S$. So we have $E.L = 2$. Write $E \sim xL + yD + z\Gamma$. From $E.L = 2$ and $E.D = 1$, we get

$$(7) \quad x = \frac{1}{3} - \frac{a}{3}z \quad \text{and} \quad y = \frac{2ma - 3d_0}{9}z - \frac{2(m-3)}{9}.$$

Inserting into $\frac{1}{2}E^2 = 0 = mx^2 - z^2 + 3xy + d_0xz + ayz$, we find

$$(8) \quad [a(ma - 3d_0) - 9]z^2 = m - 6.$$

If $m = 4$, we get from (8) that $z = \pm 1$ and $a(4a - 3d_0) = 7$. Since $d_0 > 0$, we must have $(d_0, a) = (9, 7)$, which is present in case (b). (From (7) we get the integer solution $(x, y, z) = (-2, 3, 1)$.)

If $m = 5$, we get from (8) that $z = \pm 1$ and $a(5a - 3d_0) = 8$. Since $d_0 > 0$, we must have $(d_0, a) = (2, 2)$, $(6, 4)$ or $(13, 8)$, which are present in case (c). (We can however check from (7) that $(2, 2)$ and $(13, 8)$ do not give any integer solutions for x and y , whereas $(6, 4)$ gives the integer solution $(x, y, z) = (-1, 2, 1)$.)

If $m = 6$, we get from (8) that either $z = 0$ or $a(2a - d_0) = 3$. In the first case, we get the absurdity $x = 1/3$ from (7), and in the latter we get the only solution $(d_0, a) = (5, 3)$, which inserted in (7) gives the absurdity $x = 1/3 - z$. We have therefore shown that L is base point free and that $\text{Cliff } L = 1$.

To show that L is ample we have to show by the Nakai criterion that there is no smooth curve E satisfying $E^2 = -2$ and $E.L = 0$.

By the Hodge index theorem again we have

$$2L^2(\pm E.D - 1) = L^2(D \pm E)^2 \leq (L.(D \pm E))^2 = 9,$$

giving $-1 \leq E.D \leq 1$. The cases $E.D = \pm 1$ are symmetric by interchanging E and $-E$, so we can restrict to treating the cases $E.D = 0$ and $E.D = 1$.

We can write $E \sim xL + yD + z\Gamma$. We get

$$(9) \quad x = -\frac{a}{3}z + \frac{E.D}{3}.$$

Combining this with $E.L = 0 = 2mx + 3y + d_0z$, we get

$$(10) \quad y = \frac{2ma - 3d_0}{9}z - \frac{2m(E.D)}{9}.$$

Now we use $\frac{1}{2}E^2 = -1 = mx^2 - z^2 + 3xy + d_0xz + ayz$, and find

$$(11) \quad [a(ma - 3d_0) - 9]z^2 = m(E.D)^2 - 9.$$

We first treat the case $E.D = 0$. We get

$$(12) \quad [a(ma - 3d_0) - 9]z^2 = -9,$$

which means that $z = \pm 1$ or $z = \pm 3$.

If $z = \pm 1$, we find from (12) that $ma - 3d_0 = 0$, and from (9) we find $x = \mp \frac{a}{3}$, whence $3|a$. If in addition $9|a$ or $m = 6$, then $(x, y, z) = \pm(-a/3, ma/9, 1)$ defines an effective divisor E with $E^2 = -2$ and $E.L = E.D = 0$.

If $z = \pm 3$, we find from (10) that $3|2ma$. But since $a(ma - 3d_0) = 8$, we get the absurdity $3|16$.

We now treat the case $E.D = 1$. Then we have

$$(13) \quad [a(ma - 3d_0) - 9]z^2 = m - 9.$$

We now divide into the three cases $m = 4, 5$, and 6 .

If $m = 4$, then (13) reads

$$(14) \quad [a(4a - 3d_0) - 9]z^2 = -5,$$

which means that $z = \pm 1$ and $a(4a - 3d_0) = 4$, in particular $a = 1, 2$ or 4 . Since $d_0 > 0$ we only get the solutions

$$(15) \quad (a, d_0) = (2, 2) \text{ or } (a, d_0) = (4, 5).$$

From (9) we get $x = \frac{1}{3}(1 - az) = \frac{1}{3}(1 \mp a)$, which means that we only have the possibilities $(x, z, a, d_0) = (1, -1, 2, 2)$ and $(x, z, a, d_0) = (-1, 1, 4, 5)$. Inserting into (10) we get

$$(16) \quad y = \frac{1}{9}[(8a - 3d_0)z - 8] = -2 \text{ and } 1,$$

respectively, whence $(x, y, z, a, d_0) = (1, -2, -1, 2, 2)$ and $(-1, 1, 1, 4, 5)$ are the only solutions.

If $m = 5$, then (13) reads

$$(17) \quad [a(5a - 3d_0) - 9]z^2 = -4,$$

which means that $z = \pm 1$ or $z = \pm 2$. If $z = \pm 1$, we have $a(5a - 3d_0) = 5$, and since $d_0 > 0$, there is no solution. So $z = \pm 2$ and $a(5a - 3d_0) = 8$. Again, since $d_0 > 0$, we only get the solutions

$$(18) \quad (a, d_0) = (2, 2), (a, d_0) = (4, 6) \text{ or } (a, d_0) = (8, 13).$$

From (9) we get $x = \frac{1}{3}(1 - az) = \frac{1}{3}(1 \mp 2a)$, which means that we only have the possibilities $(x, z, a, d_0) = (-1, 2, 2, 2)$, $(3, -2, 4, 6)$ or $(-5, 2, 8, 13)$. Inserting into (10) we get

$$(19) \quad y = \frac{1}{9}[(10a - 3d_0)z - 10] = 2, -6 \text{ or } 8,$$

respectively, whence $(x, y, z, a, d_0) = (-1, 2, 2, 2, 2)$, $(3, -6, -2, 4, 6)$ and $(-5, 8, 2, 8, 13)$ are the only solutions.

If $m = 6$, then (13) reads

$$(20) \quad [a(6a - 3d_0) - 9]z^2 = 3z^2[a(2a - d_0) - 3] = -3.$$

We obtain $z^2[a(2a - d_0) - 3] = -1$, which gives $z = \pm 1$ and $a(2a - d_0) = 2$. Since $d_0 > 0$, the latter yields $(a, d_0) = (2, 3)$. If $z = 1$, then (9) gives the absurdity $x = \frac{-1}{3}$. For $z = -1$ we obtain $(x, y) = (1, -3)$ from (9) and (10). Hence $(x, y, z, a, d_0) = (1, -3, -1, 2, 3)$ is the only solution for $m = 6$.

So we have proved that L is ample except for the cases (a)-(d).

It is well-known (see e.g. [SD] or [Kn1]) that an ample line bundle with $\text{Cliff } L = 1$ is very ample.

If L is ample, it follows from Proposition 5.1 that $|D|$ and $|L - D|$ are base point free and $h^1(L - D) = h^1(L - 2D) = 0$. \square

We get the corresponding statement for H :

Lemma 5.5. *Assume n , d and g does not satisfy any of the following conditions:*

- (i) $na = 3d$, with $9 \nmid a$ if $n \equiv 1, 2 \pmod{3}$, and $3 \mid a$ if $n \equiv 0 \pmod{3}$.
- (ii) $n \equiv 0 \pmod{3}$, $a = 2$ and $d = 3 + \frac{2}{3}(n - 6)$,
- (iii) $n \equiv 1 \pmod{3}$ and $d = d_0 + \frac{2}{3}(n - 4)$, for $(d_0, a) = (2, 2)$, $(5, 4)$ or $(9, 7)$,
- (iv) $n \equiv 2 \pmod{3}$ and $d = d_0 + \frac{2}{3}(n - 5)$, for $(d_0, a) = (2, 2)$, $(6, 4)$ or $(13, 8)$.

Then H is very ample and $\text{Cliff } H = 1$. Moreover, $h^1(H - iD) = 0$ for all $i \geq 0$ such that $H - iD \geq 0$.

Denote by T the scroll defined by D . Then T is smooth and of maximally balanced scroll type.

Proof. The first two statements are clear since $D.H = 3$ and $H = L + \lfloor \frac{n-4}{3} \rfloor D$, with $\text{Cliff } L = 1$ and D nef, since the cases (i), (ii), (iii) and (iv) are direct translations of the cases (a), (d), (b) and (c), respectively, in Lemma 5.4 above.

We now prove that $h^1(H - iD) = 0$ for all $i \geq 0$ such that $H - iD \geq 0$. By Lemma 5.4 we have that $h^1(L - D) = h^1(L - 2D) = 0$. Since D is nef we have $h^1(L + iD) = 0$ for all $i \geq 0$. Now let $R := L - 2D$. Then $R^2 = -4, -2$ or 0 , corresponding to $L^2 = 8, 10$ or 12 . Since we have $h^1(R) = 0$ we get $h^0(R) = 0, 1$ or 2 respectively. In the first case we are therefore done, and in the second, we clearly have $h^0(R - D) = 0$, and now we also want to show this for $L^2 = 12$. Assume that $h^0(R - D) > 0$. Then, we have

$$R = D + \Delta,$$

where $|D|$ is the moving part of $|R|$ and Δ is the fixed part. Since $R.D = 3$, we get $\Delta.D = 3$, and since $D^2 = 0$ we get $\Delta^2 = -6$. Moreover $\Delta.L = (L - 3D).L = 3$. This yields

$$\Delta = \Gamma_1 + \Gamma_2 + \Gamma_3,$$

where the Γ_i 's are smooth non-intersecting rational curves satisfying $\Gamma_i.L = \Gamma_i.D = 1$ for $i = 1, 2, 3$. Writing $\Gamma_i = x_i L + y_i D + z_i \Gamma$, the

three equations $\Gamma_i.L = 1$, $\Gamma_i.D = 1$ and $\Gamma_i^2 = -2$ yield at most two integer solutions (x_i, y_i, z_i) , whence at least two of the Γ_i s have to be equal, a contradiction.

So we have proved that $h^1(H - iD) = 0$ for all $i \geq 0$ such that $H - iD \geq 0$. By Lemma 5.2 it follows that the scroll T is of maximally balanced type, whence smooth. \square

In the next section we will describe under what conditions Γ is a smooth rational curve. We end this section with two helpful lemmas.

Lemma 5.6. *Assume L is ample. Let $\Delta \sim xL + yD + z\Gamma$ be a divisor on S such that $\Delta^2 = -2$ and set $\delta' := |\text{disc}(L, D, \Delta)|$.*

Then $\delta' = z^2\delta$, and is zero if and only if $\Delta \sim L - 2D$ and $m = 5$.

Proof. It is an easy computation to show that $\delta' = z^2\delta$. Hence it is zero if and only if $z = 0$. It is then an easy exercise to find that $\Delta \sim L - 2D$ and $m = 5$. \square

Lemma 5.7. *Let $B := 3L - mD$. (We have $B^2 = 0$ and $B.D = 9$, whence by Riemann-Roch $B > 0$.) If Δ is a smooth rational curve satisfying $\Delta^2 = -2$ and $\Delta.B \leq 0$, then we only have the following possibilities:*

$$\begin{aligned} m = 4 \quad \text{and} \quad (\Delta.L, \Delta.D, \Delta.B) &= (1, 1, -1), (4, 3, 0), (4, 4, -4), \\ &\quad (5, 4, -1), (6, 5, -2), (8, 6, 0), (9, 7, -1) \\ m = 5 \quad \text{and} \quad (\Delta.L, \Delta.D, \Delta.B) &= (1, 1, -2), (3, 2, -1), (4, 3, -3) \\ m = 6 \quad \text{and} \quad (\Delta.L, \Delta.D, \Delta.B) &= (1, 1, -3), (2, 1, 0), (3, 2, -3), (4, 2, 0). \end{aligned}$$

Proof. Set as before $R := L - 2D$. We have $3\Delta.R = \Delta.(3L - 6D) \leq \Delta.(3L - mD) \leq 0$, whence $\Delta.R \leq 0$, with equality only if $m = 6$.

If equality occurs, we have $\Delta < R$ (since $R^2 = 0$ and $R > 0$ by Proposition 5.1), whence we have a nontrivial effective decomposition $R \sim \Delta + \Delta_0$. Since $R.L = 6$ and L is ample, we have $\Delta.L \leq 5$, whence $(\Delta.L, \Delta.D) = (4, 2)$ and $(2, 1)$ are the only possibilities.

If $\Delta = R$, then $R^2 = -2$, whence $m = 5$ and $(\Delta.L, \Delta.D, \Delta.B) = (4, 3, -3)$.

So for the rest of the proof, we can assume that $\Delta.R < 0$ with $\Delta \neq R$.

If $R^2 = -2$ or 0 (i.e. $m = 5$ or 6), then $R > 0$ by Proposition 5.1, whence $\Delta < R$. If $\Delta.R \leq -2$, we get a nontrivial effective decomposition $R \sim \Delta + \Delta_0$ with $\Delta.\Delta_0 \leq 0$. But this contradicts Proposition 5.1. So $\Delta.R = -1$. Since $R.L = 4$ and 6 for $m = 5$ and 6 respectively, and L is ample, we have $\Delta.L \leq 3$ and ≤ 5 respectively. If $m = 6$

and $\Delta.L = 5$, we get $\Delta.D = 3$ and we calculate $|\text{disc}(L, D, \Delta)| = 0$, contradicting Lemma 5.6. This leaves us with the possibilities listed above for $m = 5$ and 6.

Now we treat the case $R^2 = -4$, i.e. $m = 4$. Then we have $h^0(R) = h^1(R) = 0$.

If $\Delta.R = -1$, then, since $\Delta.B = 3\Delta.R + 2\Delta.D = -3 + 2\Delta.D$ and D is nef, we get the only possibility $(\Delta.L, \Delta.D) = (1, 1)$.

If $\Delta.R \leq -2$, we get $(R - \Delta)^2 \geq -2$, whence by Riemann-Roch either $R - \Delta > 0$ or $\Delta - R > 0$. In the first case we get the contradiction $R > \Delta > 0$, so we must have $L - 2D < \Delta < 3L - 4D$ (the latter inequality due to the fact that $B^2 = 0$, $B > 0$ and $\Delta.B \leq 0$). Since L is ample, we therefore get

$$(21) \quad 3 \leq \Delta.L \leq 11,$$

and from the Hodge index theorem

$$16(-\Delta.B - 1) = (B - \Delta)^2 L^2 \leq ((B - \Delta).L)^2 = (12 - \Delta.L)^2,$$

that is

$$(22) \quad -\Delta.B \leq \lfloor \frac{(12 - \Delta.L)^2}{16} + 1 \rfloor \leq \lfloor \frac{(12 - 3)^2}{16} + 1 \rfloor = 6.$$

If $\Delta.B = 0$, then $\Delta.D = \frac{3\Delta.L}{4}$, which means by (21) that $(\Delta.L, \Delta.D) = (4, 3)$ or $(8, 6)$.

If $\Delta.B = -1$, then $\Delta.D = \frac{3\Delta.L+1}{4}$, which means by (21) that $(\Delta.L, \Delta.D) = (5, 4)$ or $(9, 7)$.

If $\Delta.B = -2$, then (21) and (22) gives $3 \leq \Delta.L \leq 8$ and $\Delta.D = \frac{3\Delta.L+2}{4}$, which means that $(\Delta.L, \Delta.D) = (6, 5)$.

Continuing this way up to $\Delta.B = -6$, one ends up with the choices listed in the lemma. \square

6. PROOF OF THEOREM 4.3

We will now complete the four steps of the proof of Theorem 4.3.

6.1. Proof of Step (I). We start with some further investigations of the $K3$ surface with the Picard lattice introduced in Lemma 5.3:

Lemma 6.1. *Assume L is ample. Then Γ is a smooth rational curve if and only if none of these special cases occur:*

- (a) $m = 4$, $3d_0 = 4a$ and $a > 9$, in which case $\Gamma \sim (3L - 4D) + (\Gamma - 3L + 4D)$ is a nontrivial effective decomposition.
- (b) $m = 5$ and $4 < d_0 < 2a$, in which case $\Gamma \sim (L - 2D) + (\Gamma - L + 2D)$ is a nontrivial effective decomposition.

- (c) $m = 6$, $d_0 = 2a$ and $a > 3$, in which case $\Gamma \sim (L - 2D) + (\Gamma - L + 2D)$ is a nontrivial effective decomposition.

Proof. Since $\Gamma^2 = -2$ and $\Gamma.H > 0$ we only need to show that Γ is irreducible. Consider $B = 3L - mD$ as defined above. Then

$$(23) \quad \delta := |\text{disc}(L, D, \Gamma)| = |2(\Gamma.D)(\Gamma.B) + 18|.$$

Case I: $\Gamma.B > 0$. Then $\delta > 18$. Assume that Γ is not irreducible. Then there has to exist a smooth rational curve $\gamma < \Gamma$ such that $\gamma.B \leq \Gamma.B$. We also have $\gamma.D \leq \Gamma.D$ by nefness of D . Set $\Delta := \Gamma - \gamma > 0$. If $\gamma.B = \Gamma.B$, then $\Delta.B = 3\Delta.L - m\Delta.D = 0$, whence $\Delta.D > 0$, since L is ample, and hence $\gamma.D < \Gamma.D$. In other words we always have $(\gamma.D)(\gamma.B) < (\Gamma.D)(\Gamma.B)$, whence

$$(24) \quad \text{disc}(L, D, \gamma) = 2(\gamma.D)(\gamma.B) + 18 < \delta.$$

If now $\gamma.B < 0$, we get from Lemma 5.7 that $(\gamma.D)(\gamma.B) \geq -16$, whence

$$(25) \quad \text{disc}(L, D, \gamma) = 2(\gamma.D)(\gamma.B) + 18 \geq -32 + 18 \geq -14 > -\delta.$$

So we must have $\text{disc}(L, D, \gamma) = 0$, whence by Lemma 5.6 we have $m = 5$ and $\gamma \sim L - 2D =: R$. By ampleness of L we must have $0 < \Delta.L = (\Gamma - L + 2D).L = d_0 - 10 + 6 = d_0 - 4$, whence $d_0 > 4$. We will now show that $d_0 < 2a$ as well, so that we end up in case (b) above.

Assume to get a contradiction that $d_0 \geq 2a$. Write $\Gamma \sim kR + \Delta_k$, for an integer $k \geq 1$ such that $\Delta_k > 0$ and $R \not\leq \Delta_k$. By our assumption we have $\Delta_k^2 = -2(k^2 + 1 + k(d_0 - 2a)) \leq -2$, so Δ_k must have at least one smooth rational curve in its support. Since we have just shown that the only smooth rational curve γ such that $\gamma.B \leq 0$ is R , we have $\gamma_0.B > 0$ for any smooth rational curve $\gamma_0 \leq \Delta_k$. Pick one such $\gamma_0 \leq \Delta_k$ such that $\gamma_0.B \leq \Delta_k.B = \Gamma.B + 3k$. Then, since also $0 \leq \gamma_0.D \leq \Delta_k.D = \Gamma.D - 3k$, and $\Gamma.B = 3d_0 - 5a \geq a = \Gamma.D$ by our assumptions, we get the contradiction

$$0 < \text{disc}(L, D, \gamma_0) = 2(\gamma_0.D)(\gamma_0.B) + 18 \leq 2(\Gamma.D - 3)(\Gamma.B + 3) + 18 < 2(\Gamma.D)(\Gamma.B) + 18 = \delta.$$

So we are in case (b). Conversely, if $m = 5$ and $4 < d_0 < 2a$, one sees that $(\Gamma - L + 2D)^2 \geq -2$ and $(\Gamma - L + 2D).L > 0$, whence by Riemann-Roch $(\Gamma - L + 2D) > 0$ and $\Gamma \sim (L - 2D) + (\Gamma - L + 2D)$ is a nontrivial effective decomposition.

Case II: $\Gamma.B = 0$. Then $\delta = 18$ and $3d_0 = ma$. Since we assume that L is ample, we have that 9 does not divide a if $m = 4$ or 5 and

3 does not divide a if $m = 6$ by Lemma 5.4. Assume that Γ is not irreducible. Then there has to exist a smooth rational curve $\gamma < \Gamma$ such that $\gamma.B \leq 0$. If $\gamma.B < 0$, then $\gamma.D > 0$ by Lemma 5.7, and we can argue as in Case I above. We end up in the case $m = 5$ and $\gamma = L - 2D$, and since $d_0 = \frac{5a}{3} < 2a$ this is a special case of (b).

So we can assume that $\gamma.B = 0$ for any smooth rational curve in the support of B . By Lemma 5.7 again, for any such γ we have the possibilities

$$(26) \quad (m, \gamma.L, \gamma.D) = (4, 4, 3), (4, 8, 6), (6, 2, 1) \text{ or } (6, 4, 2),$$

whence the case $m = 5$ is ruled out. To prove that we end up in the cases (a) and (c) above, we have to show that $a \neq 3, 6$ when $m = 4$ and $a \neq 1, 2$ when $m = 6$.

So assume $m = 4$ and $(d_0, a) = (4, 3)$ or $(8, 6)$. Since $\gamma.L < \Gamma.L = d$, we see from (26) that $(d_0, a) = (8, 6)$ and $(\gamma.L, \gamma.D) = (4, 3)$ for any $\gamma < \Gamma$. For any such γ , consider $\Delta := \Gamma - \gamma > 0$. Then $(\Delta.L, \Delta.D) = (4, 3)$. If $\Delta^2 \geq 2$, we get the contradiction from the Hodge index theorem:

$$64 = 8L^2 \leq L^2(\Delta + D)^2 \leq (L.(\Delta + D))^2 = 49.$$

If $\Delta^2 = 0$, we write $\Delta \sim xL + yD + z\Gamma$ and use the three equations

$$\begin{aligned} \Delta.L &= 8x + 3y + 8z &= 4, \\ \Delta.D &= 3x + 6z &= 3, \\ \Delta^2 &= 8x^2 - 2z^2 + 6xy + 16xz + 12yz &= 0. \end{aligned}$$

to find the absurdity $(x, y, z) = (1, -4/3, 0)$.

So $\Delta^2 \leq -2$, which means that $\Delta^2 = -2$, since for any smooth rational curve γ_0 in its support we must have $(\gamma_0.L, \gamma_0.D) = (4, 3)$. But then $\Delta.\Gamma = \gamma.\Gamma = -1$, and Δ and γ have the same intersection numbers with all three generators of $\text{Pic } S$. But then $\gamma = \Delta$ and Γ would be divisible, a contradiction.

Assume now $m = 6$ and $(d_0, a) = (2, 1)$ or $(4, 2)$. As above we end up with the only possibility $(d_0, a) = (4, 2)$ and $(\gamma.L, \gamma.D) = (2, 1)$. Now also $(\Delta.L, \Delta.D) = (2, 1)$, and $\Delta^2 = -2$ and we reach the same contradiction as above.

So we have proved that we end up in cases (a) and (c) above. Conversely, if $m = 4$ and d_0 and a satisfy the conditions in (a), one easily checks that $\Gamma \sim (3L - 4D) + (\Gamma - 3L + 4D)$ is a nontrivial effective decomposition, since both the components have self-intersection ≥ -2

and positive intersection with L . The same holds for the decomposition $\Gamma \sim (L - 2D) + (\Gamma - L + 2D)$ if $m = 6$ and d_0 and a satisfy the conditions in (c).

Case III: $\Gamma.B < 0$. As in the proof of Lemma 5.7 we have $\Gamma.R < 0$. If $m = 4$ and $\Gamma.R = -1$, we again end up with the possibility $(\Gamma.L, \Gamma.D) = (1, 1)$, in which case Γ is irreducible.

In all other cases, we have $(R - \Gamma)^2 \geq -2$, whence by Riemann-Roch either $R - \Gamma > 0$ or $\Gamma - R > 0$.

Case III(a): $R > \Gamma$. We must have $m = 5$ or 6 , since $h^0(R) = 0$ if $m = 4$ by Proposition 5.1. We proceed as in the proof of Lemma 5.7, with Γ in the place of Δ , and show that $\Gamma.R = -1$, which gives the cases:

$$(m, d_0, a) = (5, 1, 1), (5, 3, 2), (5, 4, 3), (6, 1, 1) \text{ or } (6, 3, 2).$$

We consider $m = 5$ first. If $d_0 = 1$, then Γ is irreducible, and if $(d_0, a) = (4, 3)$, we get $\delta = 0$, whence the absurdity $\Gamma \sim L - 2D$ by Lemma 5.6. So we must have $(d_0, a) = (3, 2)$ and $\Gamma.B = -1$. If Γ is reducible, there exists a smooth rational curve $\gamma < \Gamma$ such that $\gamma.B < 0$. Since $\gamma.L < \Gamma.L = 3$, we get by looking at the list in Lemma 5.7 that $(\gamma.L, \gamma.D, \gamma.B) = (1, 1, -2)$. Since then $\text{disc}(L, D, \gamma) = \text{disc}(L, D, \Gamma) = 14$, we get from Lemma 5.6 that $\gamma \sim xL + yD + z\Gamma$, for $z = \pm 1$. Using $\gamma.L = 1 = 10x + 3y + 3z$ and $\gamma.D = 1 = 3x + 2z$, we find the integer solution $(x, y, z) = (1, -2, -1)$, which yields $\Delta^2 = -6$, where $\Delta := \Gamma - \gamma$ as usual. However $\Delta.L = 2$, whence Δ has at most two components, contradicting its self-intersection number.

We next consider $m = 6$. If $d_0 = 1$, then Γ is irreducible, so we must have $(d_0, a) = (3, 2)$ and $\Gamma.B = -3$. This yields $\delta = 6$. If Γ is reducible, there exists a smooth rational curve $\gamma < \Gamma$ such that $\gamma.B < 0$. Since $\gamma.L < \Gamma.L = 3$, we get by looking at the list in Lemma 5.7 that $(\gamma.L, \gamma.D, \gamma.B) = (1, 1, -3)$ or $(2, 1, 0)$, yielding respectively $\text{disc}(L, D, \gamma) = 12$ or 18 , contradicting Lemma 5.6, which gives $z^2 = 2$ or 3 respectively.

Case III(b): $\Gamma > R$. We have

$$-\frac{9}{a} < 3d_0 - ma = \Gamma.B < 0,$$

and $(\Gamma - R).D = a - 3 \geq 0$, whence

$$(27) \quad 3 \leq a \leq 8 \quad \text{and} \quad \frac{ma}{3} - \frac{3}{a} < d_0 < \frac{ma}{3}.$$

We leave it to the reader to verify that there are no integer solutions to (27) for $m = 6$ and that the only solutions for $m = 4$ and 5 are

$$\begin{aligned} m = 4 : \quad & (d_0, a) = (5, 4), (9, 7), \\ m = 5 : \quad & (d_0, a) = (6, 4), (8, 5), (13, 8). \end{aligned}$$

The cases with $m = 5$ belong to case (b) above. We now show that we can rule out the cases with $m = 4$.

Assume Γ is reducible. Then there has to exist a smooth rational curve $\gamma < \Gamma$ such that $\gamma.B < 0$, and we can use Lemma 5.7 again.

Assume first $(d_0, a) = (5, 4)$, which gives $\delta = 10$. Then $\gamma.L < 5$, whence by Lemma 5.7 we get the possibilities $(\gamma.L, \gamma.D, \gamma.B) = (1, 1, -1)$ or $(4, 4, -4)$, yielding respectively $\text{disc}(L, D, \gamma) = 16$ or 14 , none of which are divisible by $\delta = 10$, a contradiction.

Assume now $(d_0, a) = (9, 7)$, which gives $\delta = 4$. Then $\gamma.L < 9$ and by Lemma 5.7 we get the possibilities $(\gamma.L, \gamma.D, \gamma.B) = (1, 1, -1)$, $(5, 4, -1)$, $(6, 5, -2)$ or $(4, 4, -4)$ yielding respectively $\text{disc}(L, D, \gamma) = 16, 10, 2$ or 14 . By Lemma 5.6 the only possibility is therefore $(\gamma.L, \gamma.D, \gamma.B) = (1, 1, -1)$ with $\gamma \sim xL + yD + z\Gamma$, for $z = \pm 2$. If $z = 2$ we get the absurdity $1 = \gamma.D = 3x + 14$. If $z = -2$, we get from the two equations $1 = \gamma.D = 3x - 14$ and $1 = \gamma.L = 8x + 3y - 18$ the solution $(x, y, z) = (5, -7, -2)$, so $\gamma \sim 5L - 7D - 2\Gamma$, which yields $\gamma.\Gamma = 0$. Since we have just shown that γ is the only smooth rational curve satisfying $\gamma < \Gamma$ and $\gamma.B < 0$, we can write:

$$\Gamma \sim k\gamma + \Delta,$$

for an integer $k \geq 1$ and $\Delta > 0$ satisfying

$$(28) \quad \gamma'.B \geq 0 \quad \text{for any smooth rational curve } \gamma' < \Delta,$$

$$(29) \quad \Delta^2 = -2(k^2 + 1),$$

$$(30) \quad \Delta.L = 9 - k, \quad \text{whence } k \leq 8,$$

$$(31) \quad \Delta.B = k - 1.$$

Now we claim that there has to exist a smooth rational curve $\gamma_0 < \Delta$ such that $\gamma_0.B = 0$. Indeed, write $\Delta = \Delta_0 + \Delta_1$, where Δ_0 is the (possibly zero) moving part of $|\Delta|$, and Δ_1 its fixed part. (Note that $\Delta_1 \neq 0$ by (29).) Then $\Delta_0^2 \geq 0$ and $\Delta_0.\Delta_1 \geq 0$, whence $\Delta_1^2 \leq \Delta^2 = -2(k^2 + 1)$. Now Δ_1 is a finite sum of smooth rational curves, and let l denote the number of such curves, counted with multiplicities. One easily finds that $\Delta_1^2 \geq -2l^2$, whence by (29)

$$l \geq \sqrt{k^2 + 1} > \Delta.B = k - 1,$$

and it follows that there is a smooth rational curve $\gamma_0 < \Delta$ such that $\gamma_0.B = 0$, as claimed. But then $\text{disc}(L, D, \gamma_0) = 18$, which is not divisible by $\delta = 4$, a contradiction. \square

Now we summarize the numerical conditions obtained in Lemmas 5.5 and 6.1. We need $d_0 > \frac{ma}{3} - \frac{3}{a}$ and want L to be very ample with $\text{Cliff } L = 1$ and such that the rational normal scroll $T \supseteq S$ defined by the pencil $|D|$ is smooth and of maximally balanced scroll type. Moreover we need Γ to be smooth and irreducible.

If $m = 4$ this is satisfied if $d_0 > \frac{4a}{3} - \frac{3}{a}$, $(d_0, a) \neq (2, 2), (5, 4), (9, 7)$ and $3d_0 \neq 4a$ when $a \geq 9$. The latter means that the tuples $(d_0, a) = (4, 3)$ and $(8, 6)$ are allowed. We therefore obtain the following values:

For $m = 4$: $(d_0, a) \in \{(4, 3), (8, 6)\}$; or

$$d_0 > \frac{4a}{3} - \frac{3}{a}, (d_0, a) \notin \{(2, 2), (5, 4), (9, 7)\} \text{ and } 3d_0 \neq 4a.$$

If $m = 5$ this is satisfied if $(d_0, a) \neq (2, 2)$ and either $d_0 \leq 4$ or $d_0 \geq 2a$ (since the cases $(d_0, a) = (6, 4)$ and $(13, 8)$ from Lemma 5.5 satisfy $4 < d_0 < 2a$ and since also $d_0 \geq 2a$ implies $d_0 > \frac{5a}{3} - \frac{3}{a}$). We also find that the only pairs (d_0, a) satisfying $\frac{5a}{3} - \frac{3}{a} < d_0 \leq 4$ and $(d_0, a) \neq (2, 2)$ are $(d_0, a) = (1, 1)$ and $(3, 2)$. We therefore obtain the following values:

For $m = 5$: $(d_0, a) \in \{(1, 1), (3, 2)\}$; or $d_0 \geq 2a$

If $m = 6$ this is satisfied if $d_0 > 2a - \frac{3}{a}$, $(d_0, a) \neq (3, 2)$ and $d_0 \neq 2a$ when $a \geq 3$. The latter means that the tuples $(d_0, a) = (2, 1)$ and $(4, 2)$ are allowed. We therefore obtain the following values:

For $m = 6$: $(d_0, a) \in \{(2, 1), (4, 2)\}$; or

$$d_0 > 2a - \frac{3}{a}, (d_0, a) \neq (3, 2), \text{ and } d_0 \neq 2a.$$

Using (3) and (4) we obtain:

Proposition 6.2. *Let $n \geq 4$, $d > 0$ and $a > 0$ be integers satisfying the following conditions:*

- (i) *If $n \equiv 0 \pmod{3}$, then either $(d, a) \in \{(n/3, 1), (2n/3, 2)\}$; or $d > \frac{na}{3} - \frac{3}{a}$, $(d, a) \neq (2n/3 - 1, 2)$ and $3d \neq na$.*
- (ii) *If $n \equiv 1 \pmod{3}$, then either $(d, a) \in \{(n, 3), (2n, 6)\}$; or $d > \frac{na}{3} - \frac{3}{a}$, $(d, a) \notin \{(2(n-1)/3, 2), ((4n-1)/3, 4), ((7n-1)/3, 7)\}$ and $3d \neq na$.*
- (iii) *If $n \equiv 2 \pmod{3}$, then either $(d, a) \in \{((n-2)/3, 1), ((2n-1)/3, 2)\}$; or $d \geq (n+1)a/3$.*

Then there exists a (smooth) $K3$ surface of degree $2n$ in \mathbf{P}^{n+1} , containing a smooth elliptic curve D of degree 3 and a smooth rational curve Γ of degree d with $D \cdot \Gamma = a$, and such that

$$\mathrm{Pic} S \simeq \mathbf{Z}H \oplus \mathbf{Z}D \oplus \mathbf{Z}\Gamma,$$

where H is the hyperplane section class. Furthermore, the rational normal scroll $T \supseteq S$ defined by the pencil $|D|$ is smooth and of maximally balanced scroll type.

We now set $g = n + 1$.

At this point, for each $g \geq 5$, we have found a 17-dimensional family of (smooth) projective $K3$ surfaces in \mathbf{P}^g (since the rank of the Picard lattices are 3), each with Clifford index 1, and with a rational curve as described on it. Moreover, for each member of the family, the associated 3-dimensional rational scroll T is of maximally balanced type. Moreover it is a standard fact that any polarized $K3$ surface S in a 3-dimensional rational normal scroll T is such that S is an anticanonical divisor of type $3\mathcal{H}_T - (g - 4)\mathcal{F}_T$ on T , where \mathcal{H}_T and \mathcal{F}_T denote the hyperplane section and the \mathbf{P}^1 -fibre of the scroll, respectively. The notation \mathcal{H} and \mathcal{F} will be used for corresponding divisors on a larger, four-dimensional scroll \mathcal{H} into which T will be embedded.

For all $g \geq 5$ a 3-dimensional rational scroll T of maximally balanced type in \mathbf{P}^g is isomorphic to one such, say T' in \mathbf{P}^c , with $c = 5, 6$, or 7 . Here $g = c + 3b$, where $c = m + 1 = 5, 6$ or 7 , and b positive. We let \mathcal{H}_T (as written above) and \mathcal{H}' be the divisors on this scroll corresponding to the hyperplane divisors on T and T' , respectively. Then $\mathcal{H}_T = \mathcal{H}' + b\mathcal{F}_T$.

Observe that: $3\mathcal{H}' - (c - 4)\mathcal{F}_T = 3\mathcal{H} - (g - 4)\mathcal{F}_T$, so that we can translate any question about sections of $3\mathcal{H}_T - (g - 4)\mathcal{F}_T$ on scrolls of maximally balanced type for $g \geq 5$ to one where $g = 5, 6$, or 7 .

6.2. Proof of Step (II). Let us perform Step (II). Assume first for simplicity $c = 5$, so $g = c + 3b$, for some non-negative b . We use so-called rolling factors coordinates (see for example [Ste]) Z_1, Z_2, Z_3 for each fibre of $T = T_S$, which is isomorphic to $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \mathcal{O}_{\mathbf{P}^1}(e_2) \oplus \mathcal{O}_{\mathbf{P}^1}(e_3))$, and (t, u) for the \mathbf{P}^1 , over which T is fibered. Then the equation of S , being a zero scheme of a section of $3\mathcal{H}_T - (g - 4)\mathcal{F}_T$ on T , is

$$Q = p_1(t, u)Z_1^3 + p_2(t, u)Z_1^2Z_2 + \cdots + p_{10}(t, u)Z_3^3 = 0.$$

Here the $p_i(t, u)$ are quadratic polynomials in (t, u) . If $c = 6$ or 7 , then the corresponding expression is:

$$Q = \sum_{i_1+i_2+i_3=3} p_{(i_1, i_2, i_3)}(t, u) Z_1^{i_1} Z_2^{i_2} Z_3^{i_3},$$

where $\deg p_{(i_1, i_2, i_3)} = 2i_1 + i_2 + i_3 - 2$ if $c = 6$, and $2i_1 + 2i_2 + i_3 - 3$, if $c = 7$. In the larger scroll $\mathcal{T} = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_4))$ the equation of S is given by the additional equation $Z_4 = 0$. Let $\mathbf{P}^1 = \text{Proj}(k[u, v])$, and look at the following two equations:

$$(32) \quad v(Q + Z_4 A) + u Q_1 = 0,$$

$$(33) \quad v Z_4 - u B = 0$$

Here Q_1 has the same form as Q , while

$$B = q_1(t, u) Z_1 + q_2(t, u) Z_2 + q_3(t, u) Z_3,$$

where $\deg q_i(t, u) = e_i - e_4$, for $i = 1, 2, 3$, and

$$A = \sum_{i,j} r_{i,j}(t, u) Z_i Z_j,$$

where $\deg r_{i,j}(t, u) = e_i + e_j - (g - 4 - e_4)$

For “small” values of $s = \frac{u}{v}$ equation (33) cuts out a 3-dimensional subscroll of \mathcal{T} , while equation (32) cuts out a “deformed” $K3$ surface within this subscroll. For $s = 0$ we get our well-known situation with S and T in \mathcal{T} . We may insert $Z_4 = sB$, obtained from equation (33) in equation (32), and then we get:

$$Q + s(BA(t, u, Z_1, Z_2, Z_3, sL) + Q_1(t, u, Z_1, Z_2, Z_3, sB)) = 0.$$

By choosing Q_1, A, B in a convenient way, we may express any $Q + sQ'$ in this way, for all Q' of the same form as Q (We can choose Q_1 not to involve Z_4 if we like). Hence we can obtain all possible deformations of the equation Q this way, that is we can obtain all possible deformations as sections of $3\mathcal{H}_T - (g-4)\mathcal{F}_T$ on T (We move T too, but in a family of isomorphic rational normal scrolls, and t, u, Z_1, Z_2, Z_3 are coordinates for all these scrolls, simultaneously). By choosing Q_1 (and B and A if we like) in a convenient way, we then deform the $K3$ surface to one with Picard lattice generated by a pair of generators L_i and D_i only, all with the same intersection matrix. This is true since a zero scheme of a general section of $3\mathcal{H}_T - (g-4)\mathcal{F}_T$ on T gives a general member of an 18-dimension family of polarized $K3$ surfaces, all having Picard lattice generated by such L_i and D_i .

6.3. Proof of Step (III). If we eliminate (u, v) from the two equations above, and thus form the union of all the deformed surfaces, for varying (u, v) we obtain a threefold V with equation:

$$(Q + Z_4 A)B + Z_4 Q_1 = 0.$$

By for example studying the description on p.3 in [Ste], one sees that V is the zero scheme of a section of $-K_{\mathcal{T}} = 4\mathcal{H} - (N - 5)\mathcal{F}$, where $N = e_1 + \dots + e_4 + 3 = g + e_4 + 1$ is the dimension of the projective space spanned by \mathcal{T} . Moreover, one argues like in [E-J-S] that for a general choice of A, B, Q_1 the threefold V is only singular at the finitely many points given by $Q = Q_1 = Z_4 = B = 0$, and that none of these points are contained in Γ . The number of points is:

$$(3\mathcal{H} - (g - 4)\mathcal{F})^2(\mathcal{H} - e_4\mathcal{F})^2 = 9\mathcal{H}^4 - (2g - 8 - 2e_4)\mathcal{H}^3\mathcal{F} = 9(g - 3) - (2g + 2e_4 - 8) = 7g - 19 - 2e_4$$

in the numerical ring of \mathcal{T} .

A helpful result, inspired from [E-J-S], is the following:

Lemma 6.3. *Let W be a rational normal scroll, and let $S = Z(Q_0, L_0)$ be a smooth codimension 2 subvariety of W , where Q_0 , and L_0 are effective divisors on W , which are sections of base point free line bundles of type $a\mathcal{H} + b\mathcal{F}$ on W . Let \mathfrak{L} be the linear system $\{Q_0 L_1 - Q_1 L_0\}$ on W , where L_1 varies through all elements of $|L_0|$ and Q_1 varies through all elements of $|Q_0|$. Then for a general element l of \mathfrak{L} , $\text{Sing}(l) = Z(Q_0, Q_1, L_0, L_1)$. In general $Z(Q_0, Q_1, L_0, L_1)$ will be of numerical type $Q_0^2 L_0^2$ on W .*

Proof. A straightforward generalization of the proof of Lemma 5.3 of [E-J-S]. In that lemma one expresses the elements of Q_0 , and L_0 as hypersurfaces in the projective space where W sits. In our case this is true locally each point. Since the proof is local, it works also here. \square

Remark 6.4. *Applying Lemma 6.3 in an obvious way in the case $W = \mathcal{T}$, it is clear that $Z(Q_0, Q_1, L_0, L_1) \cap W = Z(Q_1, L_1) \cap S$ will intersect Γ in an empty set for general Q_1, L_1 , since Q_0 , and L_0 are base point free divisors. Hence $Z(l)$ will contain Γ and be non-singular at all points of C for general Q_1, L_1 .*

Remark 6.5. *If the scroll type is (s, s, s, s) , $(s+1, s, s, s)$, or $(s+1, s+1, s, s)$ then both of the linear systems $3\mathcal{H} - (g - 4)\mathcal{F}$ and $\mathcal{H} - e_4\mathcal{F}$ are base point free, and Lemma 6.3 and Remark 6.4 can be applied directly. In the cases $(s+1, s+1, s+1, s)$ and $(s+1, s, s, s)$ the system $\mathcal{H} - e_4\mathcal{F}$ is base point free, while $3\mathcal{H} - (g - 4)\mathcal{F}$ has the fourth directrix*

curve (with $(Z_1, Z_2, Z_3, Z_4) = (0, 0, 0, 1)$) as base locus. A refined local study reveals that a general section of $(Q + Z_4 A)B + Z_4 Q_1$ is smooth at all points of this directrix curve simultaneously, and then an argument similar to Lemma 6.3 applies in these two cases also, and one concludes that $Q = Q_1 = Z_4 = B = 0$ is finite for general $Q_1, B(, A)$. Remark 6.4 also applies even if the Q have base points along the fourth directrix curve, since Z_4 is never zero along this curve. In the cases where \mathcal{T} is maximally balanced, but not among the 5 families of scroll types that give an in general smooth anti-canonical divisor, the singular locus of such a divisor will be the fourth directrix curve. This curve does not intersect Γ , and hence a suitably revised (essentially the same) argument applies.

Just like in the proofs of Lemmas 2.4 and 2.7 of [Og], or in Theorem 4.3 and part 5.1 of [E-J-S], we see that for an arbitrary such deformation of Q the curve Γ is isolated in W . The essential argument is given already on p. 22-23 in [Cl1].

6.4. Proof of Step (IV). This follows as on p. 25-26 in [Cl1], or as in the proof of Theorem 3.4 of [E-J-S]. Let $M = M_{d,a} = \{[C] | C \text{ has bidegree } (d, a)\}$, and let G be the parameter space of “hypersurfaces” of type $4\mathcal{H} - (N-5)\mathcal{F}$ in \mathcal{T} , where $N = e_1 + \dots + e_4 + 3$ is the dimension of the projective space \mathbf{P}^N spanned by \mathcal{T} . Study the incidence $I = I_{d,a} = \{([C], [F]) \in M \times G | C \subset F\}$. Then one easily shows:

Lemma 6.6. *Every component of I has dimension at least $\dim G$, and $\dim G = 104$ if $4e_4 - (N-5) \geq -2$.*

Proof. Let t, u, Z_1, \dots, Z_4 as usual be coordinates of \mathcal{T} . Let r, s be homogeneous coordinates of the $C = \mathbf{P}^1$, which is mapped into \mathcal{T} as a rational curve of bidegree (d, a) . This map corresponds to some (not uniquely defined) parametrization

$$t = T(r, s), u = U(r, s), Z_1 = S_1(r, s), \dots, Z_4 = S_4(r, s).$$

Here $\deg T = \deg U = a$, and $\deg S_i = d - ae_i$, for $i = 1, 2, 3, 4$. (A set of coordinates for \mathbf{P}^n are of the form $Z_i^j = Z_i t^j u^{e_i-j}$. Each coordinate is then of degree d in the variables r, s . See for example [Ste, p. 3], or [Re]). The 6-tuples

$$(T(r, s), U(r, s), S_1(r, s), \dots, S_4(r, s))$$

depend on $2(a+1) + (d-ae_1+1) + \dots + (d-ae_4+1) = 4d + a(5-N) + 6$ variables. Let $\mathcal{N} = \mathcal{N}_{d,a}$ be the set of such 6-tuples. (An open subset of) \mathcal{N} can be viewed as a parameter space for “Parametrized rational

curves of bidegree (d, a) in \mathcal{T} . The parameter space M can be viewed as a quotient of \mathcal{N} . Likewise $J = J_{d,a}$ is defined as the corresponding incidence in $\mathcal{N} \times G$, and can be viewed as a quotient of I . The fibres of these quotients have dimension $5 = \dim PGL(2) + 2$ (We have two multiplicative factors, one for (T, U) , and one for (S_1, \dots, S_4)). Hence $\dim M = 4d + a(5 - N) + 1$, and $\dim I = 4d + a(5 - N) + 1 + \dim G$ ($= 4d + a(5 - N) + 105$ if \mathcal{T} is of reasonably well balanced type). Study the incidence $R = \{(P, [F]) \in \mathcal{T} \times G \mid P \in F\}$. Then R has an equation, which is setting equal to zero a sum of monomials of type:

$$p_{i_1, \dots, i_4}(t, u) Z_1^{i_1} \cdots Z_4^{i_4}.$$

The $\dim G + 1$ coefficients of the p_{i_1, \dots, i_4} can be viewed as homogeneous coordinates of G . In this equation we now insert the parametrizations $T(r, s), U(r, s), S_1(r, s), \dots, S_4(r, s)$. This gives an equation involving the coordinates of G and the $\dim \mathcal{N}$ coefficients of these 6-tuples, and in addition r, s . We may view this as a homogeneous polynomial of degree $4d + a(5 - N)$ in r, s , since $\deg p_{i_1, \dots, i_4}(t, u) = i_1 e_1 + \cdots + i_4 e_4 - (N - 5)$. The equation of the incidence J in $\mathcal{N} \times G$ is obtained by setting all the $4d + a(5 - N) + 1$ coefficients of this polynomial equal to zero. Since this number of coefficients is equal to $\dim M$, we get $\dim M$ equations in an ambient space $\mathcal{N} \times G$ of dimension $\dim M + \dim G + 6$. Hence all components of J have dimension at least $\dim G + 6$, and consequently all components of I have dimension at least $\dim G$. Now we simply differentiate the second projection map $\pi_2 : I \rightarrow G$. If the kernel of the tangent space map is zero at a point of I , then the tangent map is injective, and therefore surjective, since $\dim I \geq \dim G$ at all points of I . Now this kernel is zero at $([\Gamma], [V])$, since $h^0(\mathcal{N}_{\Gamma/V}) = 0$, since Γ is isolated in V . Hence the tangent map is injective and surjective in an open neighborhood of $([\Gamma], [V])$ on a component of I of dimension $\dim G$, and the conclusion about the existence of an isolated rational curve of degree d holds for a general section of $4\mathcal{H} - (N - 5)\mathcal{F}$. \square

At this point the proof of Theorem 4.3 is complete.

REFERENCES

- [A-C-G-H] Arbarello, E., Cornalba, M., Griffiths, P. A., Harris, J. *Geometry of Algebraic Curves, Volume I*, number 267 in *Grundlehren der Mathematischen Wissenschaften*, Springer Verlag, Berlin, Heidelberg, New York, Tokyo (1985).
- [Ar] Artin, M. *Some numerical criteria for contractability of curves on algebraic surfaces*, American Journal of Math. **84**, 485-496 (1962).

- [B-C-K-S] Batyrev, V., Ciocan-Fontanine, I., Kim, B. van Straten, D. *Conifold Transitions and Mirror Symmetry for Calabi-Yau Intersections in Grassmannians*, Preprint alg-geom/9710022.
- [Cl1] Clemens, H., *Homological equivalence, modulo algebraic equivalence, is not finitely generated*, Publ. Math. IHES **58**, 19-38 (1983).
- [Cl2] Clemens, H., *Curves on higher-dimensional complex projective manifolds*, Proc. International Cong. Math., Berkeley, 634-40 (1986).
- [Ei-Ha] Eisenbud, D. and Harris, J., *On Varieties of Minimal Degree (A Centennial Account)*, Proceedings of Symposia in Pure Mathematics **46**, 3-13 (1987).
- [E-J-S] Ekedahl, T., Johnsen, T., Sommervoll, D.E., *Isolated rational curves on $K3$ -fibered Calabi-Yau threefolds*, Manuscripta Math. **99**, 111-33 (1999).
- [Gr-La] Green, M. and Lazarsfeld, R., *Special divisors on curves on a $K3$ surface*, Invent. Math. **89**, 357-370 (1987).
- [G-L-P] Gruson, L., Lazarsfeld, R., Peskine, C. *On a theorem of Castelnuovo and the equations defining space curves*, Invent. Math. **72**, 491-506 (1983).
- [Ha] Harris, J., *A Bound on the Geometric Genus of Projective Varieties*, Ann. Scuola Norm. Sup. Pisa Cl. Sci, **8** (4), 35-68 (1981).
- [Jo-Kn] Johnsen, T. and Knutsen, A. L., *$K3$ projective models in scrolls*, math.AG/0108183 (2001).
- [Jo-Kl1] Johnsen, T. and Kleiman, S., *Rational curves of degree at most 9 on a general quintic threefold*, Comm. Alg. **24**(8), 2721-53 (1996).
- [Jo-Kl2] Johnsen, T. and Kleiman, S., *Toward Clemens' conjecture in degrees between 10 and 24*, Serdica Math. J. **23**, 131-142 (1997).
- [Jor] Jordanger, L., *Om glatte rasjonale kurver på Calabi-Yau'ar*, Cand.sci. thesis, University of Bergen (1999).
- [Ka] Katz, S. *On the finiteness of rational curves on quintic threefolds*, Compositio Math. **60**, 151-62, (1986).
- [Kl1] Kley, H., *On the existence of curves in K -trivial threefolds*, alg-geom/981109 (1999).
- [Kl2] Kley, H., *Rigid curves in complete intersection Calabi-Yau threefolds*, Compositio Math. **123**(2), 185-208, (2000).
- [Kn1] Knutsen, A. L., *On k th order embeddings of $K3$ surfaces and Enriques surfaces*, Manuscripta Math. **104**, 211-237, (2001).
- [Kn2] Knutsen, A. L., *Smooth curves in families of Calabi-Yau threefolds*, math.AG/0110220 (2001).
- [Kn3] Knutsen, A. L., *Smooth curves on projective $K3$ surfaces*, to appear in Math. Scand.
- [Ma] Martens, H.H., *On the varieties of special divisors on a curve, II*, J. reine und angew. Math. **233**, 89-100 (1968).
- [Mori] Mori, S., *On degrees and genera of curves on smooth quartic surfaces in \mathbf{P}^3* , Nagoya Math. J. **96**, 127-32 (1984).
- [Morr] Morrison, D. R., *On $K3$ surfaces with large Picard number*, Invent. Math. **75**, 105-121 (1984).
- [Ni] Nikulin, V., *Integral symmetric bilinear forms and some of their applications*, Math. USSR-Izv. **14**, 103-167 (1980).

- [Nj] Nijse, P., *Clemens' conjecture for octic and nonic curves*, Indag. Math.(N.S.) **6(2)**, 213-21 (1995).
- [Og] Oguiso, K., *Two remarks on Calabi-Yau Moishezon threefolds*, J. Reine und Angew. Math **452**, 153-61 (1994).
- [Os] Osland, T. M., *Rational Curves in Grassmann Varieties*, Cand.sci. thesis, University of Bergen (2001).
- [Re] Reid, M., *Chapters on Algebraic Surfaces*, In: Complex algebraic geometry (Park City 1993), IAS/Park City Math., Ser. **3**, 3-159 (1997).
- [SD] Saint-Donat, B., *Projective models of K3 Surfaces*, Amer. J. Math. **96**, 602-639 (1974).
- [Sc] Schreyer, F. O., *Syzgies of canonical curves and special linear series*, Math. Ann. **275**, 105-137 (1986).
- [Se1] Segre, C., *Sulle rigate razionali in uno spazio lineare qualunque*, Atti della R. Acc. delle scienze di Torino, **XIX**, 265-282 (1883-4).
- [Se2] Segre, C., *Sulle varietà a tre dimensioni composte di serie semplici razionali di piani*, Atti della R. Acc. delle scienze di Torino **XXI**, 95-115 (1885-6).
- [Si] Sidman, J., *On the Castelnuovo-Mumford regularity of products of ideal sheaves*, math.AG/0110184 (2001).
- [Ste] Stevens, J., *Rolling factors deformations and extensions of canonical curves*, math.AG/0006189 (2000).
- [Str] Strømme, S. A., *On parametrized rational curves in Grassmann varieties*, In: "Space Curves" (Rocca di Papa 1985), Springer Lecture Notes in Mathematics, **1266**, 251-272 (1987).
- [Va] Vainsencher, I., *Enumeration of n -fold tangent hyperplanes to a surface*, Jour. of Alg. Geom. **4**, 503-26 (1995).